

! CHAPTER 2

CONCATENATION;

! Remark that concatenation is defined for everything and not just finite sequences. The advantage of this approach is that the definition makes no pre-suppositions. i

! 1. i

$\mathbb{D} \quad \wedge ; (F^{\wedge}G) ; ; (F \sqcup ((\Theta(\lambda F)) \circ G))$ i

! 2. i

$\vdash \forall F \forall G \forall H (\theta F \ \& \ F \equiv G \Rightarrow (F^{\wedge}H) \equiv (G^{\wedge}H))$ i

F, G, H , ! 1 (Prem) i

$\theta F \ \& \ F \equiv G$, ! 2 (Prem) i

$F \equiv G$, ! 3 (&E: 2) i

$(F \equiv G \Rightarrow (F \sqcup ((\Theta(\lambda F)) \circ H)) \equiv (G \sqcup ((\Theta(\lambda F)) \circ H)))$
, ! 4 (\forall E: III2.13) i

$F \equiv G \Rightarrow (F \sqcup ((\Theta(\lambda F)) \circ H)) \equiv (G \sqcup ((\Theta(\lambda F)) \circ H))$
, ! 5 ({}E: 4) i

$(F \sqcup ((\Theta(\lambda F)) \circ H)) \equiv (G \sqcup ((\Theta(\lambda F)) \circ H))$
, ! 6 (\Rightarrow E: 3,5) i

$(\theta F \ \& \ F \equiv G \Rightarrow (\lambda F) = (\lambda G))$, ! 7 (\forall E: C1.34) i

$\theta F \ \& \ F \equiv G \Rightarrow (\lambda F) = (\lambda G)$, ! 8 ({}E: 7) i

$(\lambda F) = (\lambda G)$, ! 9 (\Rightarrow E: 2,8) i

$(F \sqcup ((\Theta(\lambda F)) \circ H)) \equiv (G \sqcup ((\Theta(\lambda G)) \circ H))$
, ! 10 (=E: 6,9) i

$(F^{\wedge}H) \equiv (G \sqcup ((\Theta(\lambda G)) \circ H))$, ! 11 (\mathbb{D} I: P1,10) i

$(F^{\wedge}H) \equiv (G^{\wedge}H)$, ! 12 (\mathbb{D} I: P1,11) i

$\theta F \ \& \ F \equiv G \Rightarrow (F^{\wedge}H) \equiv (G^{\wedge}H)$, ! 13 (\Rightarrow I: 2,12) i

$(\theta F \ \& \ F \equiv G \Rightarrow (F^{\wedge}H) \equiv (G^{\wedge}H))$, ! 14 ({}I: 13) i

$\forall F \forall G \forall H (\theta F \ \& \ F \equiv G \Rightarrow (F^{\wedge}H) \equiv (G^{\wedge}H))$! 15 (\forall I: 1,14) i

\square

! 3. i

$\vdash \forall F \forall G \forall H (\theta G \ \& \ F \equiv G \Rightarrow (F^{\wedge}H) \equiv (G^{\wedge}H))$ i

F, G, H	,! 1 (Prem)	i
$\theta G \ \& \ F \equiv G$,! 2 (Prem)	i
$F \equiv G$,! 3 (&E: 2)	i
$(\theta G \ \& \ F \equiv G \Rightarrow \theta F)$,! 4 (\forall E: C1.9)	i
$\theta G \ \& \ F \equiv G \Rightarrow \theta F$,! 5 ((\Rightarrow)E: 4)	i
θF	,! 6 (\Rightarrow E: 2,5)	i
$\theta F \ \& \ F \equiv G$,! 7 (&I: 3,6)	i
$(\theta F \ \& \ F \equiv G \Rightarrow (F \wedge H) \equiv (G \wedge H))$,! 8 (\forall E: P2)	i
$\theta F \ \& \ F \equiv G \Rightarrow (F \wedge H) \equiv (G \wedge H)$,! 9 ((\Rightarrow)E: 8)	i
$(F \wedge H) \equiv (G \wedge H)$,! 10 (\Rightarrow E: 2,9)	i
$\theta G \ \& \ F \equiv G \Rightarrow (F \wedge H) \equiv (G \wedge H)$,! 11 (\Rightarrow I: 2,10)	i
$(\theta G \ \& \ F \equiv G \Rightarrow (F \wedge H) \equiv (G \wedge H))$,! 12 ((\Rightarrow)I: 11)	i
$\forall F \forall G \forall H (\theta G \ \& \ F \equiv G \Rightarrow (F \wedge H) \equiv (G \wedge H))$! 13 (\forall I: 1,12)	i

□

! 4.

$\vdash \forall F \forall G \forall H (F \equiv G \Rightarrow (H \wedge F) \equiv (H \wedge G))$ i

F, G, H	,! 1 (Prem)	i
$F \equiv G$,! 2 (Prem)	i
$(F \equiv G \Rightarrow ((\Theta(\lambda H)) \circ F) \equiv ((\Theta(\lambda H)) \circ G))$,! 3 (\forall E: III10.8)	i
$F \equiv G \Rightarrow ((\Theta(\lambda H)) \circ F) \equiv ((\Theta(\lambda H)) \circ G)$,! 4 ((\Rightarrow)E: 3)	i
$((\Theta(\lambda H)) \circ F) \equiv ((\Theta(\lambda H)) \circ G)$,! 5 (\Rightarrow E: 2,4)	i
$(((\Theta(\lambda H)) \circ F) \equiv ((\Theta(\lambda H)) \circ G) \Rightarrow (H \sqcup ((\Theta(\lambda H)) \circ F)) \equiv (H \sqcup ((\Theta(\lambda H)) \circ G)))$,! 6 (\forall E: III2.14)	i
$((\Theta(\lambda H)) \circ F) \equiv ((\Theta(\lambda H)) \circ G)$		
$\Rightarrow (H \sqcup ((\Theta(\lambda H)) \circ F)) \equiv (H \sqcup ((\Theta(\lambda H)) \circ G))$,! 7 ((\Rightarrow)E: 6)	i
$(H \sqcup ((\Theta(\lambda H)) \circ F)) \equiv (H \sqcup ((\Theta(\lambda H)) \circ G))$		

,! 8 (\Rightarrow E: 5,7) i

$$(H^{\wedge}F) \equiv (H \sqcup ((\Theta(\lambda H)) \circ G))$$

,! 9 (\mathbb{D} I: P1,8) i

$$(H^{\wedge}F) \equiv (H^{\wedge}G)$$

,! 10 (\mathbb{D} I: P1,9) i

$$F \equiv G \Rightarrow (H^{\wedge}F) \equiv (H^{\wedge}G)$$

,! 11 (\Rightarrow I: 2,10) i

$$(F \equiv G \Rightarrow (H^{\wedge}F) \equiv (H^{\wedge}G))$$

,! 12 ($(\)$ I: 11) i

$$\forall F \forall G \forall H (F \equiv G \Rightarrow (H^{\wedge}F) \equiv (H^{\wedge}G))$$

! 13 (\forall I: 1,12) i

□

! 5.

$\vdash \forall F \forall G \forall H \forall I (\Theta F \ \& \ F \equiv G \ \& \ I \equiv (F^{\wedge}H) \Rightarrow I \equiv (G^{\wedge}H))$ i

F, G, H, I ,! 1 (Prem) i

$$\Theta F \ \& \ F \equiv G \ \& \ I \equiv (F^{\wedge}H)$$

,! 2 (Prem) i

$$\Theta F \ \& \ F \equiv G$$

,! 3 ($\&$ E: 2) i

$$I \equiv (F^{\wedge}H)$$

,! 4 ($\&$ E: 2) i

$$(\Theta F \ \& \ F \equiv G \Rightarrow (F^{\wedge}H) \equiv (G^{\wedge}H))$$

,! 5 (\forall E: P2) i

$$\Theta F \ \& \ F \equiv G \Rightarrow (F^{\wedge}H) \equiv (G^{\wedge}H)$$

,! 6 ($(\)$ I: 5) i

$$(F^{\wedge}H) \equiv (G^{\wedge}H)$$

,! 7 (\Rightarrow E: 3,6) i

$$I \equiv (F^{\wedge}H) \ \& \ (F^{\wedge}H) \equiv (G^{\wedge}H)$$

,! 8 ($\&$ I: 4,7) i

$$(I \equiv (F^{\wedge}H) \ \& \ (F^{\wedge}H) \equiv (G^{\wedge}H) \Rightarrow I \equiv (G^{\wedge}H))$$

,! 9 (\forall E: III1.15) i

$$I \equiv (F^{\wedge}H) \ \& \ (F^{\wedge}H) \equiv (G^{\wedge}H) \Rightarrow I \equiv (G^{\wedge}H)$$

,! 10 ($(\)$ E: 9) i

$$I \equiv (G^{\wedge}H)$$

,! 11 (\Rightarrow E: 8,10) i

$$\Theta F \ \& \ F \equiv G \ \& \ I \equiv (F^{\wedge}H) \Rightarrow I \equiv (G^{\wedge}H)$$

,! 12 (\Rightarrow I: 2,11) i

$$(\Theta F \ \& \ F \equiv G \ \& \ I \equiv (F^{\wedge}H) \Rightarrow I \equiv (G^{\wedge}H))$$

,! 13 ($(\)$ I: 12) i

$$\forall F \forall G \forall H \forall I (\Theta F \ \& \ F \equiv G \ \& \ I \equiv (F^{\wedge}H) \Rightarrow I \equiv (G^{\wedge}H))$$

! 14 (\forall I: 1,13) i

□

! 6.

$\vdash \forall F \forall G \forall H \forall I (\Theta F \ \& \ F \equiv G \ \& \ (F^{\wedge}H) \equiv I \Rightarrow (G^{\wedge}H) \equiv I)$ i

F, G, H, I	,! 1 (Prem)	i
$\theta F \ \& \ F \equiv G \ \& \ (F^H) \equiv I$,! 2 (Prem)	i
$\theta F \ \& \ F \equiv G$,! 3 (&E: 2)	i
$(F^H) \equiv I$,! 4 (&E: 2)	i
$(\theta F \ \& \ F \equiv G \Rightarrow (F^H) \equiv (G^H))$,! 5 (\forall E: P2)	i
$\theta F \ \& \ F \equiv G \Rightarrow (F^H) \equiv (G^H)$,! 6 (()I: 5)	i
$(F^H) \equiv (G^H)$,! 7 (\Rightarrow E: 3,6)	i
$(F^H) \equiv (G^H) \ \& \ (F^H) \equiv I$,! 8 (&I: 4,7)	i
$((F^H) \equiv (G^H) \ \& \ (F^H) \equiv I \Rightarrow (G^H) \equiv I)$,! 9 (\forall E: III1.18)	i
$(F^H) \equiv (G^H) \ \& \ (F^H) \equiv I \Rightarrow (G^H) \equiv I$,! 10 (()E: 9)	i
$(G^H) \equiv I$,! 11 (\Rightarrow E: 8,10)	i
$\theta F \ \& \ F \equiv G \ \& \ (F^H) \equiv I \Rightarrow (G^H) \equiv I$,! 12 (\Rightarrow I: 2,11)	i
$(\theta F \ \& \ F \equiv G \ \& \ (F^H) \equiv I \Rightarrow (G^H) \equiv I)$,! 13 (()I: 12)	i
$\forall F \forall G \forall H \forall I (\theta F \ \& \ F \equiv G \ \& \ (F^H) \equiv I \Rightarrow (G^H) \equiv I)$! 14 (\forall I: 1,13)	i

□

! 7.

$\vdash \forall F \forall G \forall H \forall I (F \equiv G \ \& \ I \equiv (H^F) \Rightarrow I \equiv (H^G))$		i
F, G, H, I	,! 1 (Prem)	i
$F \equiv G \ \& \ I \equiv (H^F)$,! 2 (Prem)	i
$F \equiv G$,! 3 (&E: 2)	i
$I \equiv (H^F)$,! 4 (&E: 2)	i
$(F \equiv G \Rightarrow (H^F) \equiv (H^G))$,! 5 (\forall E: P4)	i
$F \equiv G \Rightarrow (H^F) \equiv (H^G)$,! 6 (()I: 5)	i
$(H^F) \equiv (H^G)$,! 7 (\Rightarrow E: 3,6)	i
$I \equiv (H^F) \ \& \ (H^F) \equiv (H^G)$,! 8 (&I: 4,7)	i
$(I \equiv (H^F) \ \& \ (H^F) \equiv (H^G) \Rightarrow I \equiv (H^G))$		i

,! 9 ($\forall E$: III1.15) ;

$I \equiv (H \wedge F) \ \& \ (H \wedge F) \equiv (H \wedge G) \Rightarrow I \equiv (H \wedge G)$
,! 10 ($(\)E$: 9) ;

$I \equiv (H \wedge G)$
,! 11 ($\Rightarrow E$: 8,10) ;

$F \equiv G \ \& \ I \equiv (H \wedge F) \Rightarrow I \equiv (H \wedge G)$
,! 12 ($\Rightarrow I$: 2,11) ;

($F \equiv G \ \& \ I \equiv (H \wedge F) \Rightarrow I \equiv (H \wedge G)$)
,! 13 ($(\)I$: 12) ;

$\forall F \forall G \forall H \forall I (F \equiv G \ \& \ I \equiv (H \wedge F) \Rightarrow I \equiv (H \wedge G))$
! 14 ($\forall I$: 1,13) ;

□

! 8. ;

$\vdash \forall F \forall G \forall H \forall I (F \equiv G \ \& \ (H \wedge F) \equiv I \Rightarrow (H \wedge G) \equiv I)$;

F, G, H, I
,! 1 (Prem) ;

$F \equiv G \ \& \ (H \wedge F) \equiv I$
,! 2 (Prem) ;

$F \equiv G$
,! 3 ($\&E$: 2) ;

$(H \wedge F) \equiv I$
,! 4 ($\&E$: 2) ;

($F \equiv G \Rightarrow (H \wedge F) \equiv (H \wedge G)$)
,! 5 ($\forall E$: P4) ;

$F \equiv G \Rightarrow (H \wedge F) \equiv (H \wedge G)$
,! 6 ($(\)I$: 5) ;

$(H \wedge F) \equiv (H \wedge G)$
,! 7 ($\Rightarrow E$: 3,6) ;

$(H \wedge F) \equiv (H \wedge G) \ \& \ (H \wedge F) \equiv I$
,! 8 ($\&I$: 4,7) ;

($(H \wedge F) \equiv (H \wedge G) \ \& \ (H \wedge F) \equiv I \Rightarrow (H \wedge G) \equiv I$)
,! 9 ($\forall E$: III1.18) ;

$(H \wedge F) \equiv (H \wedge G) \ \& \ (H \wedge F) \equiv I \Rightarrow (H \wedge G) \equiv I$
,! 10 ($(\)E$: 9) ;

$(H \wedge G) \equiv I$
,! 11 ($\Rightarrow E$: 8,10) ;

$F \equiv G \ \& \ (H \wedge F) \equiv I \Rightarrow (H \wedge G) \equiv I$
,! 12 ($\Rightarrow I$: 2,11) ;

($F \equiv G \ \& \ (H \wedge F) \equiv I \Rightarrow (H \wedge G) \equiv I$)
,! 13 ($(\)I$: 12) ;

$\forall F \forall G \forall H \forall I (F \equiv G \ \& \ (H \wedge F) \equiv I \Rightarrow (H \wedge G) \equiv I)$
! 14 ($\forall I$: 1,13) ;

□

! 9. ;

$\vdash \forall F \forall G (\theta F \ \& \ \theta G \Rightarrow ((F \wedge G)^D) \equiv (1 _ ((\lambda F)+(\lambda G))))$ i
F, G ,! 1 (Prem) i
 $\theta F \ \& \ \theta G$,! 2 (Prem) i
 $((F \sqcup ((\Theta(\lambda F)) \circ G))^D) \equiv ((F^D) \cup (((\Theta(\lambda F)) \circ G)^D))$
, ! 3 ($\forall E$: III5.19) i
 $((F \wedge G)^D) \equiv ((F^D) \cup (((\Theta(\lambda F)) \circ G)^D))$
, ! 4 ($\mathbb{D}I$: P1,3) i
 $(\theta F \ \& \ \theta G \Rightarrow ((F^D) \cup (((\Theta(\lambda F)) \circ G)^D)) \equiv (1 _ ((\lambda F)+(\lambda G))))$
, ! 5 ($\forall E$: C1.57) i
 $\theta F \ \& \ \theta G \Rightarrow ((F^D) \cup (((\Theta(\lambda F)) \circ G)^D)) \equiv (1 _ ((\lambda F)+(\lambda G)))$
, ! 6 ($()E$: 5) i
 $((F^D) \cup (((\Theta(\lambda F)) \circ G)^D)) \equiv (1 _ ((\lambda F)+(\lambda G)))$
, ! 7 ($\Rightarrow E$: 2,6) i
 $((F \wedge G)^D) \equiv ((F^D) \cup (((\Theta(\lambda F)) \circ G)^D))$
 $\& ((F^D) \cup (((\Theta(\lambda F)) \circ G)^D)) \equiv (1 _ ((\lambda F)+(\lambda G)))$
, ! 8 ($\&I$: 4,7) i
 $(((F \wedge G)^D) \equiv ((F^D) \cup (((\Theta(\lambda F)) \circ G)^D))$
 $\& ((F^D) \cup (((\Theta(\lambda F)) \circ G)^D)) \equiv (1 _ ((\lambda F)+(\lambda G)))$
 $\Rightarrow ((F \wedge G)^D) \equiv (1 _ ((\lambda F)+(\lambda G)))$
, ! 9 ($\forall E$: III1.15) i
 $((F \wedge G)^D) \equiv ((F^D) \cup (((\Theta(\lambda F)) \circ G)^D))$
 $\& ((F^D) \cup (((\Theta(\lambda F)) \circ G)^D)) \equiv (1 _ ((\lambda F)+(\lambda G)))$
 $\Rightarrow ((F \wedge G)^D) \equiv (1 _ ((\lambda F)+(\lambda G)))$
, ! 10 ($()E$: 9) i
 $((F \wedge G)^D) \equiv (1 _ ((\lambda F)+(\lambda G)))$, ! 11 ($\Rightarrow E$: 8,10) i
 $\theta F \ \& \ \theta G \Rightarrow ((F \wedge G)^D) \equiv (1 _ ((\lambda F)+(\lambda G)))$
, ! 12 ($\Rightarrow I$: 2,11) i
 $(\theta F \ \& \ \theta G \Rightarrow ((F \wedge G)^D) \equiv (1 _ ((\lambda F)+(\lambda G)))$
, ! 13 ($()I$: 12) i
 $\forall F \forall G (\theta F \ \& \ \theta G \Rightarrow ((F \wedge G)^D) \equiv (1 _ ((\lambda F)+(\lambda G))))$
! 14 ($\forall I$: 1,13) i
 \square
! 10. i

$\vdash \forall F \forall G (\theta F \ \& \ \theta G \Rightarrow f (F \wedge G))$		i
F, G	,! 1 (Prem)	i
$\theta F \ \& \ \theta G$,! 2 (Prem)	i
θF	,! 3 (&E: 2)	i
θG	,! 4 (&E: 2)	i
$(\theta F \Rightarrow f F)$,! 5 (\forall E: C1.2)	i
$\theta F \Rightarrow f F$,! 6 (()E: 5)	i
f F	,! 7 (\Rightarrow E: 3,6)	i
$(\theta F \Rightarrow \omega[(\lambda F)])$,! 8 (\forall E: C1.21)	i
$\theta F \Rightarrow \omega[(\lambda F)]$,! 9 (()E: 8)	i
$\omega[(\lambda F)]$,! 10 (\Rightarrow E: 3,9)	i
$\theta G \ \& \ \omega[(\lambda F)]$,! 11 (&I: 4,10)	i
$(\theta G \ \& \ \omega[(\lambda F)] \Rightarrow f ((\Theta(\lambda F)) \circ G))$,! 12 (\forall E: C1.15; (λF): C1.17,3)	i
$\theta G \ \& \ \omega[(\lambda F)] \Rightarrow f ((\Theta(\lambda F)) \circ G)$,! 13 (()E: 12)	i
f (($\Theta(\lambda F)$) \circ G)	,! 14 (\Rightarrow E: 11,13)	i
f F & f (($\Theta(\lambda F)$) \circ G)	,! 15 (&I: 7,14)	i
$(\theta F \ \& \ \theta G \Rightarrow ((F^D) \cap (((\Theta(\lambda F)) \circ G)^D)) \equiv \phi)$,! 16 (\forall E: C1.58)	i
$\theta F \ \& \ \theta G \Rightarrow ((F^D) \cap (((\Theta(\lambda F)) \circ G)^D)) \equiv \phi$,! 17 (()E: 16)	i
$((F^D) \cap (((\Theta(\lambda F)) \circ G)^D)) \equiv \phi$,! 18 (\Rightarrow E: 2,17)	i
f F & f (($\Theta(\lambda F)$) \circ G) & ((F^D) \cap ((($\Theta(\lambda F)$) \circ G)^D) $\equiv \phi$,! 19 (&I: 15,18)	i
$(f F \ \& \ f ((\Theta(\lambda F)) \circ G) \ \& \ ((F^D) \cap (((\Theta(\lambda F)) \circ G)^D)) \equiv \phi$ $\Rightarrow f (F \sqcup ((\Theta(\lambda F)) \circ G))$,! 20 (\forall E: III8.7)	i
f F & f (($\Theta(\lambda F)$) \circ G) & ((F^D) \cap ((($\Theta(\lambda F)$) \circ G)^D) $\equiv \phi$ $\Rightarrow f (F \sqcup ((\Theta(\lambda F)) \circ G))$,! 21 (()E: 20)	i

$f (F \sqcup ((\Theta(\lambda F)) \circ G))$,!	22 ($\Rightarrow E$: 19,21)	i
$f (F \wedge G)$,!	23 ($\mathbb{D}I$: P1,22)	i
$\theta F \ \& \ \theta G \Rightarrow f (F \wedge G)$,!	24 ($\Rightarrow I$: 2,23)	i
$(\theta F \ \& \ \theta G \Rightarrow f (F \wedge G))$,!	25 ($(\)I$: 24)	i
$\forall F \forall G (\theta F \ \& \ \theta G \Rightarrow f (F \wedge G))$!	26 ($\forall I$: 1,25)	i
\square			
! 11.			i
$\vdash \forall F \forall G (\theta F \ \& \ \theta G \Rightarrow (\lambda(F \wedge G)) = ((\lambda F) + (\lambda G)))$			i
F, G	,!	1 (Prem)	i
$\theta F \ \& \ \theta G$,!	2 (Prem)	i
θF	,!	3 ($\&E$: 2)	i
θG	,!	4 ($\&E$: 2)	i
$(\theta F \ \& \ \theta G \Rightarrow ((F \wedge G)^D) \equiv (1 _ ((\lambda F) + (\lambda G))))$,!	5 ($\forall E$: P9)	i
$\theta F \ \& \ \theta G \Rightarrow ((F \wedge G)^D) \equiv (1 _ ((\lambda F) + (\lambda G)))$,!	6 ($(\)E$: 5)	i
$((F \wedge G)^D) \equiv (1 _ ((\lambda F) + (\lambda G)))$,!	7 ($\Rightarrow E$: 2,6)	i
$(\theta F \ \& \ \theta G \Rightarrow f (F \wedge G))$,!	8 ($\forall E$: P10)	i
$\theta F \ \& \ \theta G \Rightarrow f (F \wedge G)$,!	9 ($(\)E$: 8)	i
$f (F \wedge G)$,!	10 ($\Rightarrow E$: 2,9)	i
$((F \wedge G)^D) \equiv (1 _ ((\lambda F) + (\lambda G))) \ \& \ f (F \wedge G)$,!	11 ($\&I$: 7,10)	i
$(\theta F \Rightarrow \omega[(\lambda F)])$,!	12 ($\forall E$: C1.21)	i
$\theta F \Rightarrow \omega[(\lambda F)]$,!	13 ($(\)E$: 12)	i
$\omega[(\lambda F)]$,!	14 ($\Rightarrow E$: 3,13)	i
$(\theta G \Rightarrow \omega[(\lambda G)])$,!	15 ($\forall E$: C1.21)	i
$\theta G \Rightarrow \omega[(\lambda G)]$,!	16 ($(\)E$: 15)	i
$\omega[(\lambda G)]$,!	17 ($\Rightarrow E$: 4,16)	i
$\omega[(\lambda F)] \ \& \ \omega[(\lambda G)]$,!	18 ($\&I$: 14,17)	i

$(\omega[(\lambda F)] \ \& \ \omega[(\lambda G)] \Rightarrow \omega[((\lambda F)+(\lambda G))])$
, ! 19 ($\forall E$: V1.8;
 (λF) : C1.17,3;
 (λG) : C1.17,4) i

$\omega[(\lambda F)] \ \& \ \omega[(\lambda G)] \Rightarrow \omega[((\lambda F)+(\lambda G))]$, ! 20 ($()E$: 19) i

$\omega[((\lambda F)+(\lambda G))]$, ! 21 ($\Rightarrow E$: 18,20) i

$((F \wedge G)^D) \equiv (1 _ ((\lambda F)+(\lambda G))) \ \& \ f(F \wedge G) \ \& \ \omega[((\lambda F)+(\lambda G))]$
, ! 22 ($\&I$: 11,21) i

$(((F \wedge G)^D) \equiv (1 _ ((\lambda F)+(\lambda G))) \ \& \ f(F \wedge G) \ \& \ \omega[((\lambda F)+(\lambda G))]$
 $\Rightarrow (\lambda(F \wedge G)) = ((\lambda F)+(\lambda G)))$
, ! 23 ($\forall E$: C1.19;
 $((\lambda F)+(\lambda G))$: V1.7,18) i

$((F \wedge G)^D) \equiv (1 _ ((\lambda F)+(\lambda G))) \ \& \ f(F \wedge G) \ \& \ \omega[((\lambda F)+(\lambda G))]$
 $\Rightarrow (\lambda(F \wedge G)) = ((\lambda F)+(\lambda G))$
, ! 24 ($()E$: 23) i

$(\lambda(F \wedge G)) = ((\lambda F)+(\lambda G))$, ! 25 ($\Rightarrow E$: 22,24) i

$\theta F \ \& \ \theta G \Rightarrow (\lambda(F \wedge G)) = ((\lambda F)+(\lambda G))$, ! 26 ($\Rightarrow I$: 2,25) i

$(\theta F \ \& \ \theta G \Rightarrow (\lambda(F \wedge G)) = ((\lambda F)+(\lambda G)))$, ! 27 ($()I$: 26) i

$\forall F \forall G (\theta F \ \& \ \theta G \Rightarrow (\lambda(F \wedge G)) = ((\lambda F)+(\lambda G)))$
! 28 ($\forall I$: 1,27) i

\square

! 12. i

$\vdash \forall F \forall G (\theta F \ \& \ \theta G \Rightarrow \leq[(\lambda F), (\lambda(F \wedge G))])$ i

F, G , ! 1 (Prem) i

$\theta F \ \& \ \theta G$, ! 2 (Prem) i

$(\theta F \ \& \ \theta G \Rightarrow (\lambda(F \wedge G)) = ((\lambda F)+(\lambda G)))$
, ! 3 ($\forall E$: P11) i

$\theta F \ \& \ \theta G \Rightarrow (\lambda(F \wedge G)) = ((\lambda F)+(\lambda G))$, ! 4 ($()E$: 3) i

$(\lambda(F \wedge G)) = ((\lambda F) + (\lambda G))$, ! 5 ($\Rightarrow E$: 2,4) i

$\omega[(\lambda F)] \ \& \ \omega[(\lambda G)]$, ! 6 ($\mathbb{T}E$: V1.7,5) i

θF , ! 7 ($\&E$: 2) i

θG , ! 8 ($\&E$: 2) i

$(\omega[(\lambda F)] \ \& \ \omega[(\lambda G)] \Rightarrow \leq[(\lambda F), ((\lambda F) + (\lambda G))])$, ! 9 ($\forall E$: V3.33;	
	(λF): C1.17,7;	
	(λG): C1.17,8)	i
$\omega[(\lambda F)] \ \& \ \omega[(\lambda G)] \Rightarrow \leq[(\lambda F), ((\lambda F) + (\lambda G))]$, ! 10 ($(())E$: 9)	i
$\leq[(\lambda F), ((\lambda F) + (\lambda G))]$, ! 11 ($\Rightarrow E$: 6,10)	i
$\leq[(\lambda F), (\lambda(F \wedge G))]$, ! 12 ($=E$: 5,11)	i
$\theta F \ \& \ \theta G \Rightarrow \leq[(\lambda F), (\lambda(F \wedge G))]$, ! 13 ($\Rightarrow I$: 2,12)	i
$(\theta F \ \& \ \theta G \Rightarrow \leq[(\lambda F), (\lambda(F \wedge G))])$, ! 14 ($(())I$: 13)	i
$\forall F \forall G (\theta F \ \& \ \theta G \Rightarrow \leq[(\lambda F), (\lambda(F \wedge G))])$! 15 ($\forall I$: 1,14)	i
\square		
! 13.		i
$\vdash \forall F \forall G (\theta F \ \& \ \theta G \Rightarrow \leq[(\lambda G), (\lambda(F \wedge G))])$		i
F, G	, ! 1 (Prem)	i
$\theta F \ \& \ \theta G$, ! 2 (Prem)	i
$(\theta F \ \& \ \theta G \Rightarrow (\lambda(F \wedge G)) = ((\lambda F) + (\lambda G)))$, ! 3 ($\forall E$: P11)	i
$\theta F \ \& \ \theta G \Rightarrow (\lambda(F \wedge G)) = ((\lambda F) + (\lambda G))$, ! 4 ($(())E$: 3)	i
$(\lambda(F \wedge G)) = ((\lambda F) + (\lambda G))$, ! 5 ($\Rightarrow E$: 2,4)	i
$\omega[(\lambda F)] \ \& \ \omega[(\lambda G)]$, ! 6 ($\mathbb{T}E$: V1.7,5)	i
θF	, ! 7 ($\&E$: 2)	i
θG	, ! 8 ($\&E$: 2)	i
$(\omega[(\lambda F)] \ \& \ \omega[(\lambda G)] \Rightarrow \leq[(\lambda G), ((\lambda F) + (\lambda G))])$, ! 9 ($\forall E$: V3.34;	
	(λF): C1.17,7;	
	(λG): C1.17,8)	i
$\omega[(\lambda F)] \ \& \ \omega[(\lambda G)] \Rightarrow \leq[(\lambda G), ((\lambda F) + (\lambda G))]$, ! 10 ($(())E$: 9)	i
$\leq[(\lambda G), ((\lambda F) + (\lambda G))]$, ! 11 ($\Rightarrow E$: 6,10)	i
$\leq[(\lambda G), (\lambda(F \wedge G))]$, ! 12 ($=E$: 5,11)	i

$\theta F \ \& \ \theta G \Rightarrow \leq[(\lambda G), (\lambda(F \wedge G))]$,! 13 (\Rightarrow I: 2,12) i
 $(\theta F \ \& \ \theta G \Rightarrow \leq[(\lambda G), (\lambda(F \wedge G))]$) ,! 14 ((I: 13) i
 $\forall F \forall G (\theta F \ \& \ \theta G \Rightarrow \leq[(\lambda G), (\lambda(F \wedge G))]$) ! 15 (\forall I: 1,14) i
 \square

! 14. i
 $\vdash \forall F \forall G (\theta F \ \& \ \theta G \Rightarrow \theta (F \wedge G))$ i
F, G ,! 1 (Prem) i
 $\theta F \ \& \ \theta G$,! 2 (Prem) i
 $(\theta F \ \& \ \theta G \Rightarrow (\lambda(F \wedge G)) = ((\lambda F) + (\lambda G)))$,! 3 (\forall E: P11) i
 $\theta F \ \& \ \theta G \Rightarrow (\lambda(F \wedge G)) = ((\lambda F) + (\lambda G))$,! 4 ((E: 3) i
 $(\lambda(F \wedge G)) = ((\lambda F) + (\lambda G))$,! 5 (\Rightarrow E: 2,4) i
 $\theta (F \wedge G)$,! 6 (\mathbb{T} E: C1.17,5) i
 $\theta F \ \& \ \theta G \Rightarrow \theta (F \wedge G)$,! 7 (\Rightarrow I: 2,6) i
 $(\theta F \ \& \ \theta G \Rightarrow \theta (F \wedge G))$,! 8 ((I: 7) i
 $\forall F \forall G (\theta F \ \& \ \theta G \Rightarrow \theta (F \wedge G))$! 9 (\forall I: 1,8) i
 \square

! 15. i
 $\vdash \forall F \forall G \forall i (\theta G \ \& \ (1 _ (\lambda F))[i] \Rightarrow ((F \wedge G)'i) = (F'i))$ i
F, G, i ,! 1 (Prem) i
 $\theta G \ \& \ (1 _ (\lambda F))[i]$,! 2 (Prem) i
 θG ,! 3 ($\&$ E: 2) i
 $(1 _ (\lambda F))[i]$,! 4 ($\&$ E: 2) i
 $(f (F \sqcup ((\Theta(\lambda F)) \circ G)) \ \& \ (F^D)[i] \Rightarrow ((F \sqcup ((\Theta(\lambda F)) \circ G))'i) = (F'i))$,! 5 (\forall E: III8.31) i
 $f (F \sqcup ((\Theta(\lambda F)) \circ G)) \ \& \ (F^D)[i] \Rightarrow ((F \sqcup ((\Theta(\lambda F)) \circ G))'i) = (F'i)$,! 6 ((E: 5) i
 $f (F \wedge G) \ \& \ (F^D)[i] \Rightarrow ((F \wedge G)'i) = (F'i)$

	,! 7 ($\mathbb{D}I$: P1,6)	i
! To show: $f(F \wedge G) \ \& \ (F^D)[i]$		i
((1 _ (λF))[i] $\Rightarrow \theta F$)	,! 8 ($\forall E$: C1.27)	i
(1 _ (λF))[i] $\Rightarrow \theta F$,! 9 ($()E$: 8)	i
θF	,! 10 ($\Rightarrow E$: 4,9)	i
$\theta F \ \& \ \theta G$,! 11 ($\&I$: 3,10)	i
($\theta F \ \& \ \theta G \Rightarrow f(F \wedge G)$)	,! 12 ($\forall E$: P10)	i
$\theta F \ \& \ \theta G \Rightarrow f(F \wedge G)$,! 13 ($()E$: 12)	i
$f(F \wedge G)$,! 14 ($\Rightarrow E$: 11,13)	i
((1 _ (λF))[i] $\Rightarrow (F^D)[i]$)	,! 15 ($\forall E$: C1.31)	i
(1 _ (λF))[i] $\Rightarrow (F^D)[i]$,! 16 ($()E$: 15)	i
$(F^D)[i]$,! 17 ($\Rightarrow E$: 4,16)	i
$f(F \wedge G) \ \& \ (F^D)[i]$,! 18 ($\&I$: 14,17)	i
$((F \wedge G)'i) = (F'i)$,! 19 ($\Rightarrow E$: 7,18)	i
$\theta G \ \& \ (1 _ (\lambda F))[i] \Rightarrow ((F \wedge G)'i) = (F'i)$,! 20 ($\Rightarrow I$: 2,19)	i
($\theta G \ \& \ (1 _ (\lambda F))[i] \Rightarrow ((F \wedge G)'i) = (F'i)$)	,! 21 ($()I$: 20)	i
$\forall F \forall G \forall i$ ($\theta G \ \& \ (1 _ (\lambda F))[i] \Rightarrow ((F \wedge G)'i) = (F'i)$)	! 22 ($\forall I$: 1,21)	i

□

! 16.

$\vdash \forall F \forall G \forall i$ ($\theta G \ \& \ \leq[1,i] \ \& \ \leq[i,(\lambda F)] \Rightarrow ((F \wedge G)'i) = (F'i)$)		i
F, G, i	,! 1 (Prem)	i
$\theta G \ \& \ \leq[1,i] \ \& \ \leq[i,(\lambda F)]$,! 2 (Prem)	i
θG	,! 3 ($\&E$: 2)	i
$\leq[1,i] \ \& \ \leq[i,(\lambda F)]$,! 4 ($\&E$: 2)	i
$\leq[i,(\lambda F)]$,! 5 ($\&E$: 2)	i
θF	,! 6 ($\mathbb{T}E$: C1.17,5)	i

$(\leq[1, i] \ \& \ \leq[i, (\lambda F)]) \Rightarrow (1 _ (\lambda F))[i]$, ! 7 ($\forall E$: VI2.4; (λF): C1.17,6)	i
$\leq[1, i] \ \& \ \leq[i, (\lambda F)] \Rightarrow (1 _ (\lambda F))[i]$, ! 8 ($(_)E$: 7)	i
$(1 _ (\lambda F))[i]$, ! 9 ($\Rightarrow E$: 4,8)	i
$\theta G \ \& \ (1 _ (\lambda F))[i]$, ! 10 ($\&I$: 3,9)	i
$(\theta G \ \& \ (1 _ (\lambda F))[i] \Rightarrow ((F \wedge G)'i) = (F'i))$, ! 11 ($\forall E$: P15)	i
$\theta G \ \& \ (1 _ (\lambda F))[i] \Rightarrow ((F \wedge G)'i) = (F'i)$, ! 12 ($(_)E$: 11)	i
$((F \wedge G)'i) = (F'i)$, ! 13 ($\Rightarrow E$: 10,12)	i
$\theta G \ \& \ \leq[1, i] \ \& \ \leq[i, (\lambda F)] \Rightarrow ((F \wedge G)'i) = (F'i)$, ! 14 ($\Rightarrow I$: 2,13)	i
$(\theta G \ \& \ \leq[1, i] \ \& \ \leq[i, (\lambda F)] \Rightarrow ((F \wedge G)'i) = (F'i))$, ! 15 ($(_)I$: 14)	i
$\forall F \forall G \forall i (\theta G \ \& \ \leq[1, i] \ \& \ \leq[i, (\lambda F)] \Rightarrow ((F \wedge G)'i) = (F'i))$! 16 ($\forall I$: 1,15)	i
\square		
! 17.		
$\vdash \forall F \forall G \forall n \forall i ((1 _ (\lambda G))[i] \ \& \ (\lambda F) = n$		
$\Rightarrow ((F \wedge G)'(n+i)) = (G'i))$		i
F, G, n, i	, ! 1 (Prem)	i
$(1 _ (\lambda G))[i] \ \& \ (\lambda F) = n$, ! 2 (Prem)	i
$(1 _ (\lambda G))[i]$, ! 3 ($\&E$: 2)	i
$(\lambda F) = n$, ! 4 ($\&E$: 2)	i
$(\lambda F) = n \Rightarrow \omega[n]$, ! 5 ($\forall E$: C1.22)	i
$(\lambda F) = n \Rightarrow \omega[n]$, ! 6 ($(_)E$: 5)	i
$\omega[n]$, ! 7 ($\Rightarrow E$: 4,6)	i
$((1 _ (\lambda G))[i] \Rightarrow \theta G)$, ! 8 ($\forall E$: C1.27)	i
$(1 _ (\lambda G))[i] \Rightarrow \theta G$, ! 9 ($(_)E$: 8)	i
θG	, ! 10 ($\Rightarrow E$: 3,9)	i

$((1 _ (\lambda G))[i] \Rightarrow \omega[i])$,! 11 ($\forall E$: VI2.22; (λG) : C1.17,10)	i
$(1 _ (\lambda G))[i] \Rightarrow \omega[i]$,! 12 ($(\)E$: 11)	i
$\omega[i]$,! 13 ($\Rightarrow E$: 3,12)	i
$\omega[n] \ \& \ \omega[i]$,! 14 ($\&I$: 7,13)	i
$(\mathbf{f} (\mathbf{F} \sqcup ((\Theta(\lambda F)) \circ \mathbf{G})) \ \& \ (((\Theta(\lambda F)) \circ \mathbf{G})^D) [(n+i)]$ $\Rightarrow ((\mathbf{F} \sqcup ((\Theta(\lambda F)) \circ \mathbf{G})) ^\prime (n+i)) = (((\Theta(\lambda F)) \circ \mathbf{G}) ^\prime (n+i)))$,! 15 ($\forall E$: III8.32; $(n+i)$: V1.7,14)	i
$\mathbf{f} (\mathbf{F} \sqcup ((\Theta(\lambda F)) \circ \mathbf{G})) \ \& \ (((\Theta(\lambda F)) \circ \mathbf{G})^D) [(n+i)]$ $\Rightarrow ((\mathbf{F} \sqcup ((\Theta(\lambda F)) \circ \mathbf{G})) ^\prime (n+i)) = (((\Theta(\lambda F)) \circ \mathbf{G}) ^\prime (n+i))$,! 16 ($(\)E$: 15)	i
$\mathbf{f} (\mathbf{F} \wedge \mathbf{G}) \ \& \ (((\Theta(\lambda F)) \circ \mathbf{G})^D) [(n+i)]$ $\Rightarrow ((\mathbf{F} \wedge \mathbf{G}) ^\prime (n+i)) = (((\Theta(\lambda F)) \circ \mathbf{G}) ^\prime (n+i))$,! 17 ($\mathbb{D}I$: P1,16)	i
$\mathbf{f} (\mathbf{F} \wedge \mathbf{G}) \ \& \ (((\Theta n) \circ \mathbf{G})^D) [(n+i)]$ $\Rightarrow ((\mathbf{F} \wedge \mathbf{G}) ^\prime (n+i)) = (((\Theta n) \circ \mathbf{G}) ^\prime (n+i))$,! 18 ($=E$: 4,17)	i
$\theta \ \mathbf{F}$,! 19 ($\mathbb{T}E$: 4,C1.17)	i
$\theta \ \mathbf{F} \ \& \ \theta \ \mathbf{G}$,! 20 ($\&I$: 10,19)	i
$(\theta \ \mathbf{F} \ \& \ \theta \ \mathbf{G} \Rightarrow \mathbf{f} (\mathbf{F} \wedge \mathbf{G}))$,! 21 ($\forall E$: P10)	i
$\theta \ \mathbf{F} \ \& \ \theta \ \mathbf{G} \Rightarrow \mathbf{f} (\mathbf{F} \wedge \mathbf{G})$,! 22 ($(\)E$: 21)	i
$\mathbf{f} (\mathbf{F} \wedge \mathbf{G})$,! 23 ($\Rightarrow E$: 20,22)	i
$(\theta \ \mathbf{G} \Rightarrow \mathbf{f} \ \mathbf{G})$,! 24 ($\forall E$: C1.2)	i
$\theta \ \mathbf{G} \Rightarrow \mathbf{f} \ \mathbf{G}$,! 25 ($(\)E$: 24)	i
$\mathbf{f} \ \mathbf{G}$,! 26 ($\Rightarrow E$: 10,25)	i
$\mathbf{f} \ \mathbf{G} \ \& \ \omega[n] \ \& \ \omega[i]$,! 27 ($\&I$: 14,26)	i
$((1 _ (\lambda G))[i] \Rightarrow (\mathbf{G}^D)[i])$,! 28 ($\forall E$: C1.31)	i
$(1 _ (\lambda G))[i] \Rightarrow (\mathbf{G}^D)[i]$,! 29 ($(\)E$: 28)	i
$(\mathbf{G}^D)[i]$,! 30 ($\Rightarrow E$: 3,29)	i

$f G \ \& \ \omega[n] \ \& \ \omega[i] \ \& \ (G^D)[i]$,! 31 (&I: 27,30) ;

($f G \ \& \ \omega[n] \ \& \ \omega[i] \ \& \ (G^D)[i]$
 $\Rightarrow (((\Theta n) \circ G) \wedge (n+i)) = (G \wedge i)$)
,! 32 ($\forall E$: V10.31) ;

$f G \ \& \ \omega[n] \ \& \ \omega[i] \ \& \ (G^D)[i]$
 $\Rightarrow (((\Theta n) \circ G) \wedge (n+i)) = (G \wedge i)$
,! 33 ($()E$: 32) ;

$((((\Theta n) \circ G) \wedge (n+i)) = (G \wedge i))$,! 34 ($\Rightarrow E$: 31,33) ;

$f ((\Theta n) \circ G) \ \& \ (((\Theta n) \circ G)^D)[(n+i)]$,! 35 ($\mathbb{T}E$: III8.20,34) ;

$((((\Theta n) \circ G)^D)[(n+i)])$,! 36 (&E: 35) ;

$f (F \wedge G) \ \& \ (((\Theta n) \circ G)^D)[(n+i)]$,! 37 (&I: 23,36) ;

$((F \wedge G) \wedge (n+i)) = (((\Theta n) \circ G) \wedge (n+i))$,! 38 ($\Rightarrow E$: 18,37) ;

$((F \wedge G) \wedge (n+i)) = (G \wedge i)$,! 39 ($=E$: 34,38) ;

$(1 _ (\lambda G))[i] \ \& \ (\lambda F) = n \Rightarrow ((F \wedge G) \wedge (n+i)) = (G \wedge i)$
,! 40 ($\Rightarrow I$: 2,39) ;

($(1 _ (\lambda G))[i] \ \& \ (\lambda F) = n \Rightarrow ((F \wedge G) \wedge (n+i)) = (G \wedge i)$)
,! 41 ($()I$: 40) ;

$\forall F \forall G \forall n \forall i ((1 _ (\lambda G))[i] \ \& \ (\lambda F) = n \Rightarrow ((F \wedge G) \wedge (n+i)) = (G \wedge i))$
! 42 ($\forall I$: 1,41) ;

□

! 18. ;

$\vdash \forall F \forall G \forall i (\theta F \ \& \ (1 _ (\lambda G))[i] \Rightarrow ((F \wedge G) \wedge ((\lambda F)+i)) = (G \wedge i))$;

F, G, i ,! 1 (Prem) ;

$\theta F \ \& \ (1 _ (\lambda G))[i]$,! 2 (Prem) ;

θF ,! 3 (&E: 2) ;

$(1 _ (\lambda G))[i]$,! 4 (&E: 2) ;

$(\lambda F) = (\lambda F)$,! 5 ($=I$;
 (λF) : C1.17,3) ;

$(1 _ (\lambda G))[i] \ \& \ (\lambda F) = (\lambda F)$,! 6 (&I: 4,5) ;

($(1 _ (\lambda G))[i] \ \& \ (\lambda F) = (\lambda F)$
 $\Rightarrow ((F \wedge G) \wedge ((\lambda F)+i)) = (G \wedge i)$)

	,! 7 ($\forall E$: P17;	
	(λF) : C1.17,3)	i
$(1 _ (\lambda G))[i] \ \& \ (\lambda F) = (\lambda F)$		
$\Rightarrow ((F \wedge G) \wedge ((\lambda F)+i)) = (G \wedge i)$		
	,! 8 ($()E$: 7)	i
$((F \wedge G) \wedge ((\lambda F)+i)) = (G \wedge i)$,! 9 ($\Rightarrow E$: 6,8)	i
$\theta F \ \& \ (1 _ (\lambda G))[i] \Rightarrow ((F \wedge G) \wedge ((\lambda F)+i)) = (G \wedge i)$		
	,! 10 ($\Rightarrow I$: 2,9)	i
$(\theta F \ \& \ (1 _ (\lambda G))[i] \Rightarrow ((F \wedge G) \wedge ((\lambda F)+i)) = (G \wedge i))$		
	,! 11 ($()I$: 10)	i
$\forall F \forall G \forall i (\theta F \ \& \ (1 _ (\lambda G))[i] \Rightarrow ((F \wedge G) \wedge ((\lambda F)+i)) = (G \wedge i))$		
	! 12 ($\forall I$: 1,11)	i
\square		
! 19.		i
$\vdash \forall F \forall G \forall i (\theta F \ \& \ \leq[1,i] \ \& \ \leq[i,(\lambda G)] \Rightarrow ((F \wedge G) \wedge ((\lambda F)+i)) = (G \wedge i))$		i
F, G, i	,! 1 (Prem)	i
$\theta F \ \& \ \leq[1,i] \ \& \ \leq[i,(\lambda G)]$,! 2 (Prem)	i
θF	,! 3 ($\&E$: 2)	i
$\leq[1,i] \ \& \ \leq[i,(\lambda G)]$,! 4 ($\&E$: 2)	i
$\leq[i,(\lambda G)]$,! 5 ($\&E$: 2)	i
θG	,! 6 ($\mathbb{T}E$: C1.17,5)	i
$\theta F \ \& \ \theta G$,! 7 ($\&I$: 3,6)	i
$(\leq[1,i] \ \& \ \leq[i,(\lambda G)] \Rightarrow (1 _ (\lambda G))[i])$		
	,! 8 ($\forall E$: VI2.4;	
	(λG) : C1.17,6)	i
$\leq[1,i] \ \& \ \leq[i,(\lambda G)] \Rightarrow (1 _ (\lambda G))[i]$,! 9 ($()E$: 8)	i
$(1 _ (\lambda G))[i]$,! 10 ($\Rightarrow E$: 4,9)	i
$\theta F \ \& \ (1 _ (\lambda G))[i]$,! 11 ($\&I$: 3,10)	i
$(\theta F \ \& \ (1 _ (\lambda G))[i] \Rightarrow ((F \wedge G) \wedge ((\lambda F)+i)) = (G \wedge i))$		
	,! 12 ($\forall E$: P18)	i
$\theta F \ \& \ (1 _ (\lambda G))[i] \Rightarrow ((F \wedge G) \wedge ((\lambda F)+i)) = (G \wedge i)$		
	,! 13 ($()E$: 12)	i

$$((\mathbf{F}^{\wedge}\mathbf{G})'((\lambda\mathbf{F})+\mathbf{i})) = (\mathbf{G}'\mathbf{i}) \quad ,! 14 (\Rightarrow\mathbf{E}: 11,13) \quad ;$$

$$\theta \mathbf{F} \ \& \ \leq[1,\mathbf{i}] \ \& \ \leq[\mathbf{i},(\lambda\mathbf{G})] \Rightarrow ((\mathbf{F}^{\wedge}\mathbf{G})'((\lambda\mathbf{F})+\mathbf{i})) = (\mathbf{G}'\mathbf{i}) \quad ,! 15 (\Rightarrow\mathbf{I}: 2,14) \quad ;$$

$$(\theta \mathbf{F} \ \& \ \leq[1,\mathbf{i}] \ \& \ \leq[\mathbf{i},(\lambda\mathbf{G})] \Rightarrow ((\mathbf{F}^{\wedge}\mathbf{G})'((\lambda\mathbf{F})+\mathbf{i})) = (\mathbf{G}'\mathbf{i})) \quad ,! 16 ((\mathbf{I}): 15) \quad ;$$

$$\forall\mathbf{F}\forall\mathbf{G}\forall\mathbf{i} (\theta \mathbf{F} \ \& \ \leq[1,\mathbf{i}] \ \& \ \leq[\mathbf{i},(\lambda\mathbf{G})] \Rightarrow ((\mathbf{F}^{\wedge}\mathbf{G})'((\lambda\mathbf{F})+\mathbf{i})) = (\mathbf{G}'\mathbf{i})) \quad ! 17 (\forall\mathbf{I}: 1,16) \quad ;$$

□

! 20. i

$$\vdash \forall\mathbf{F}\forall\mathbf{G}\forall\mathbf{i} (\theta \mathbf{G} \ \& \ <[(\lambda\mathbf{F}),\mathbf{i}] \ \& \ \leq[\mathbf{i},(\lambda(\mathbf{F}^{\wedge}\mathbf{G}))]) \Rightarrow ((\mathbf{F}^{\wedge}\mathbf{G})'\mathbf{i}) = (\mathbf{G}'(\mathbf{i}-(\lambda\mathbf{F}))) \quad ;$$

F,G,i ,! 1 (Prem) i

$\theta \mathbf{G} \ \& \ <[(\lambda\mathbf{F}),\mathbf{i}] \ \& \ \leq[\mathbf{i},(\lambda(\mathbf{F}^{\wedge}\mathbf{G}))]$,! 2 (Prem) i

$\theta \mathbf{G}$,! 3 (&\mathbf{E}: 2) i

$<[(\lambda\mathbf{F}),\mathbf{i}]$,! 4 (&\mathbf{E}: 2) i

$\leq[\mathbf{i},(\lambda(\mathbf{F}^{\wedge}\mathbf{G}))]$,! 5 (&\mathbf{E}: 2) i

$\theta \mathbf{F}$,! 6 (\mathbf{T}\mathbf{E}: C1.17,4) i

$\theta \mathbf{F} \ \& \ \theta \mathbf{G}$,! 7 (&\mathbf{I}: 3,6) i

$(<[(\lambda\mathbf{F}),\mathbf{i}] \Rightarrow \leq[((\lambda\mathbf{F})+1),\mathbf{i}])$,! 8 (\forall\mathbf{E}: V4.36; (\lambda\mathbf{F}): C1.17,6) i

$<[(\lambda\mathbf{F}),\mathbf{i}] \Rightarrow \leq[((\lambda\mathbf{F})+1),\mathbf{i}]$,! 9 ((\mathbf{E}): 8) i

$\leq[((\lambda\mathbf{F})+1),\mathbf{i}]$,! 10 (\Rightarrow\mathbf{E}: 4,9) i

$(\leq[((\lambda\mathbf{F})+1),\mathbf{i}] \Rightarrow \leq[1,(\mathbf{i}-(\lambda\mathbf{F}))])$,! 11 (\forall\mathbf{E}: V6.84; (\lambda\mathbf{F}): C1.17,6) i

$\leq[((\lambda\mathbf{F})+1),\mathbf{i}] \Rightarrow \leq[1,(\mathbf{i}-(\lambda\mathbf{F}))]$,! 12 ((\mathbf{E}): 11) i

$\leq[1,(\mathbf{i}-(\lambda\mathbf{F}))]$,! 13 (\Rightarrow\mathbf{E}: 10,12) i

$\theta \mathbf{F} \ \& \ \leq[1,(\mathbf{i}-(\lambda\mathbf{F}))]$,! 14 (&\mathbf{I}: 6,13) i

$(\theta \mathbf{F} \ \& \ \theta \mathbf{G} \Rightarrow (\lambda(\mathbf{F}^{\wedge}\mathbf{G})) = ((\lambda\mathbf{F})+(\lambda\mathbf{G})))$,! 15 (\forall\mathbf{E}: P11) i

$\theta \mathbf{F} \ \& \ \theta \mathbf{G} \Rightarrow (\lambda(\mathbf{F}^{\wedge}\mathbf{G})) = ((\lambda\mathbf{F})+(\lambda\mathbf{G}))$,! 16 ((\mathbf{E}): 15) i

$(\lambda(\mathbf{F}^{\wedge}\mathbf{G})) = ((\lambda\mathbf{F})+(\lambda\mathbf{G}))$,! 17 ($\Rightarrow\mathbf{E}$: 7,16) ;
 $\leq[\mathbf{i},((\lambda\mathbf{F})+(\lambda\mathbf{G}))]$,! 18 ($=\mathbf{E}$: 5,17) ;
 $\leq[(\lambda\mathbf{F}),\mathbf{i}]$,! 19 ($\mathbf{T}\mathbf{E}$: v5.7,13) ;
 $\leq[\mathbf{i},((\lambda\mathbf{F})+(\lambda\mathbf{G}))] \ \& \ \leq[(\lambda\mathbf{F}),\mathbf{i}]$,! 20 ($\&\mathbf{I}$: 18,19) ;
 $(\leq[\mathbf{i},((\lambda\mathbf{F})+(\lambda\mathbf{G}))] \ \& \ \leq[(\lambda\mathbf{F}),\mathbf{i}] \Rightarrow \leq[(\mathbf{i}-(\lambda\mathbf{F})),(\lambda\mathbf{G})])$
,! 21 ($\forall\mathbf{E}$: v6.83;
 $(\lambda\mathbf{F})$: C1.17,6;
 $(\lambda\mathbf{G})$: C1.17,3) ;
 $\leq[\mathbf{i},((\lambda\mathbf{F})+(\lambda\mathbf{G}))] \ \& \ \leq[(\lambda\mathbf{F}),\mathbf{i}] \Rightarrow \leq[(\mathbf{i}-(\lambda\mathbf{F})),(\lambda\mathbf{G})]$
,! 22 ($(\)\mathbf{E}$: 21) ;
 $\leq[(\mathbf{i}-(\lambda\mathbf{F})),(\lambda\mathbf{G})]$,! 23 ($\Rightarrow\mathbf{E}$: 20,22) ;
 $\theta \mathbf{F} \ \& \ \leq[1,(\mathbf{i}-(\lambda\mathbf{F}))] \ \& \ \leq[(\mathbf{i}-(\lambda\mathbf{F})),(\lambda\mathbf{G})]$
,! 24 ($\&\mathbf{I}$: 14,23) ;
 $(\theta \mathbf{F} \ \& \ \leq[1,(\mathbf{i}-(\lambda\mathbf{F}))] \ \& \ \leq[(\mathbf{i}-(\lambda\mathbf{F})),(\lambda\mathbf{G})]$
 $\Rightarrow ((\mathbf{F}^{\wedge}\mathbf{G})'((\lambda\mathbf{F})+(\mathbf{i}-(\lambda\mathbf{F})))) = (\mathbf{G}'(\mathbf{i}-(\lambda\mathbf{F})))$
,! 25 ($\forall\mathbf{E}$: p19;
 $(\mathbf{i}-(\lambda\mathbf{F}))$: v5.7,19) ;
 $\theta \mathbf{F} \ \& \ \leq[1,(\mathbf{i}-(\lambda\mathbf{F}))] \ \& \ \leq[(\mathbf{i}-(\lambda\mathbf{F})),(\lambda\mathbf{G})]$
 $\Rightarrow ((\mathbf{F}^{\wedge}\mathbf{G})'((\lambda\mathbf{F})+(\mathbf{i}-(\lambda\mathbf{F})))) = (\mathbf{G}'(\mathbf{i}-(\lambda\mathbf{F})))$
,! 26 ($(\)\mathbf{E}$: 25) ;
 $((\mathbf{F}^{\wedge}\mathbf{G})'((\lambda\mathbf{F})+(\mathbf{i}-(\lambda\mathbf{F})))) = (\mathbf{G}'(\mathbf{i}-(\lambda\mathbf{F})))$
,! 27 ($\Rightarrow\mathbf{E}$: 24,26) ;
 $(\leq[(\lambda\mathbf{F}),\mathbf{i}] \Rightarrow ((\lambda\mathbf{F})+(\mathbf{i}-(\lambda\mathbf{F}))) = \mathbf{i})$,! 28 ($\forall\mathbf{E}$: v6.4;
 $(\lambda\mathbf{F})$: C1.17,6) ;
 $\leq[(\lambda\mathbf{F}),\mathbf{i}] \Rightarrow ((\lambda\mathbf{F})+(\mathbf{i}-(\lambda\mathbf{F}))) = \mathbf{i}$,! 29 ($(\)\mathbf{E}$: 28) ;
 $((\lambda\mathbf{F})+(\mathbf{i}-(\lambda\mathbf{F}))) = \mathbf{i}$,! 30 ($\Rightarrow\mathbf{E}$: 19,29) ;
 $((\mathbf{F}^{\wedge}\mathbf{G})'\mathbf{i}) = (\mathbf{G}'(\mathbf{i}-(\lambda\mathbf{F})))$,! 31 ($=\mathbf{E}$: 27,30) ;
 $\theta \mathbf{G} \ \& \ <[(\lambda\mathbf{F}),\mathbf{i}] \ \& \ \leq[\mathbf{i},(\lambda(\mathbf{F}^{\wedge}\mathbf{G}))] \Rightarrow ((\mathbf{F}^{\wedge}\mathbf{G})'\mathbf{i}) = (\mathbf{G}'(\mathbf{i}-(\lambda\mathbf{F})))$
,! 32 ($\Rightarrow\mathbf{I}$: 2,31) ;
 $(\theta \mathbf{G} \ \& \ <[(\lambda\mathbf{F}),\mathbf{i}] \ \& \ \leq[\mathbf{i},(\lambda(\mathbf{F}^{\wedge}\mathbf{G}))]$
 $\Rightarrow ((\mathbf{F}^{\wedge}\mathbf{G})'\mathbf{i}) = (\mathbf{G}'(\mathbf{i}-(\lambda\mathbf{F})))$
,! 33 ($(\)\mathbf{I}$: 32) ;
 $\forall\mathbf{F}\forall\mathbf{G}\forall\mathbf{i} (\theta \mathbf{G} \ \& \ <[(\lambda\mathbf{F}),\mathbf{i}] \ \& \ \leq[\mathbf{i},(\lambda(\mathbf{F}^{\wedge}\mathbf{G}))]$
 $\Rightarrow ((\mathbf{F}^{\wedge}\mathbf{G})'\mathbf{i}) = (\mathbf{G}'(\mathbf{i}-(\lambda\mathbf{F})))$
! 34 ($\forall\mathbf{I}$: 1,33) ;

$\leq[i, (\lambda F)]$,!	20 (&E: 18)	i
$(\leq[1, i] \Rightarrow \omega[i])$,!	21 ($\forall E$: V3.7)	i
$\leq[1, i] \Rightarrow \omega[i]$,!	22 ($(\)E$: 21)	i
$\omega[i]$,!	23 ($\Rightarrow E$: 19,22)	i
$\omega[i] \ \& \ \omega[(\lambda G)]$,!	24 (&I: 8,23)	i
$(\omega[i] \ \& \ \omega[(\lambda G)] \Rightarrow \leq[i, (\lambda G)] \vee <[(\lambda G), i])$,!	25 ($\forall E$: V4.43; (λG): C1.17,10)	i
$\omega[i] \ \& \ \omega[(\lambda G)] \Rightarrow \leq[i, (\lambda G)] \vee <[(\lambda G), i]$,!	26 ($(\)E$: 25)	i
$\leq[i, (\lambda G)] \vee <[(\lambda G), i]$,!	27 ($\Rightarrow E$: 24,26)	i
$\leq[i, (\lambda G)]$,!	28 (Prem)	i
$\leq[1, i] \ \& \ \leq[i, (\lambda G)]$,!	29 (&I: 19,28)	i
$(\leq[1, i] \ \& \ \leq[i, (\lambda G)] \Rightarrow (F'i) = (G'i))$,!	30 ($\forall E$: 4)	i
$\leq[1, i] \ \& \ \leq[i, (\lambda G)] \Rightarrow (F'i) = (G'i)$,!	31 ($(\)E$: 30)	i
$(F'i) = (G'i)$,!	32 ($\Rightarrow E$: 29,31)	i
$\theta H \ \& \ \leq[1, i]$,!	33 (&I: 11,19)	i
$\theta H \ \& \ \leq[1, i] \ \& \ \leq[i, (\lambda G)]$,!	34 (&I: 28,33)	i
$(\theta H \ \& \ \leq[1, i] \ \& \ \leq[i, (\lambda G)] \Rightarrow ((G^H)'i) = (G'i))$,!	35 ($\forall E$: P16)	i
$\theta H \ \& \ \leq[1, i] \ \& \ \leq[i, (\lambda G)] \Rightarrow ((G^H)'i) = (G'i)$,!	36 ($(\)E$: 35)	i
$((G^H)'i) = (G'i)$,!	37 ($\Rightarrow E$: 34,36)	i
$(F'i) = ((G^H)'i)$,!	38 (=E: 32,37)	i
$\leq[i, (\lambda G)] \Rightarrow (F'i) = ((G^H)'i)$,!	39 ($\Rightarrow I$: 28,38)	i
$<[(\lambda G), i]$,!	40 (Prem)	i
$\theta H \ \& \ <[(\lambda G), i]$,!	41 (&I: 11,40)	i
$\leq[i, (\lambda(G^H))]$,!	42 (=E: 16,20)	i
$\theta H \ \& \ <[(\lambda G), i] \ \& \ \leq[i, (\lambda(G^H))]$			

,! 43 (&I: 41,42) ;

$$\begin{aligned} & (\theta H \ \& \ <[(\lambda G), i] \ \& \ \leq[i, (\lambda(G^H))]) \\ & \Rightarrow ((G^H)'i) = (H'(i - (\lambda G))) \end{aligned}$$

,! 44 ($\forall E$: P20) ;

$$\begin{aligned} & \theta H \ \& \ <[(\lambda G), i] \ \& \ \leq[i, (\lambda(G^H))]) \\ & \Rightarrow ((G^H)'i) = (H'(i - (\lambda G))) \end{aligned}$$

,! 45 (()E: 44) ;

$$((G^H)'i) = (H'(i - (\lambda G)))$$

,! 46 ($\Rightarrow E$: 43,45) ;

$$<[(\lambda G), i] \ \& \ \leq[i, (\lambda F)]$$

,! 47 (&I: 20,40) ;

$$(\ <[(\lambda G), i] \ \& \ \leq[i, (\lambda F)] \Rightarrow (F'i) = (H'(i - (\lambda G))))$$

,! 48 ($\forall E$: 5) ;

$$<[(\lambda G), i] \ \& \ \leq[i, (\lambda F)] \Rightarrow (F'i) = (H'(i - (\lambda G)))$$

,! 49 (()E: 48) ;

$$(F'i) = (H'(i - (\lambda G)))$$

,! 50 ($\Rightarrow E$: 47,49) ;

$$(F'i) = ((G^H)'i)$$

,! 51 (=E: 46,50) ;

$$<[(\lambda G), i] \Rightarrow (F'i) = ((G^H)'i)$$

,! 52 ($\Rightarrow I$: 40,51) ;

$$(F'i) = ((G^H)'i)$$

,! 53 ($\forall E$: 27,39,52) ;

$$\leq[1, i] \ \& \ \leq[i, (\lambda F)] \Rightarrow (F'i) = ((G^H)'i)$$

,! 54 ($\Rightarrow I$: 18,53) ;

$$(\ \leq[1, i] \ \& \ \leq[i, (\lambda F)] \Rightarrow (F'i) = ((G^H)'i))$$

,! 55 (()I: 54) ;

$$\forall i (\ \leq[1, i] \ \& \ \leq[i, (\lambda F)] \Rightarrow (F'i) = ((G^H)'i))$$

,! 56 ($\forall I$: 17,55) ;

$$(\lambda F) = (\lambda(G^H))$$

$$\& \ \forall i (\ \leq[1, i] \ \& \ \leq[i, (\lambda F)] \Rightarrow (F'i) = ((G^H)'i))$$

,! 57 (&I: 16,56) ;

$$(\lambda F) = (\lambda(G^H))$$

$$\& \ \forall i (\ \leq[1, i] \ \& \ \leq[i, (\lambda F)] \Rightarrow (F'i) = ((G^H)'i))$$

$$\Rightarrow F \equiv (G^H)$$

,! 58 ($\forall E$: C1.42) ;

$$(\lambda F) = (\lambda(G^H))$$

$$\& \ \forall i (\ \leq[1, i] \ \& \ \leq[i, (\lambda F)] \Rightarrow (F'i) = ((G^H)'i))$$

$$\Rightarrow F \equiv (G^H)$$

,! 59 (()E: 58) ;

$$F \equiv (G^H)$$

,! 60 ($\Rightarrow E$: 57,59) ;

$(\lambda F) = ((\lambda G) + (\lambda H))$
 $\& \forall i (\leq[1, i] \& \leq[i, (\lambda G)] \Rightarrow (F' i) = (G' i))$
 $\& \forall i (\leq[(\lambda G), i] \& \leq[i, (\lambda F)] \Rightarrow (F' i) = (H' (i - (\lambda G))))$
 $\Rightarrow F \equiv (G \wedge H)$
, ! 61 (\Rightarrow I: 2, 60) ;

$((\lambda F) = ((\lambda G) + (\lambda H))$
 $\& \forall i (\leq[1, i] \& \leq[i, (\lambda G)] \Rightarrow (F' i) = (G' i))$
 $\& \forall i (\leq[(\lambda G), i] \& \leq[i, (\lambda F)] \Rightarrow (F' i) = (H' (i - (\lambda G))))$
 $\Rightarrow F \equiv (G \wedge H))$
, ! 62 ($()$ I: 61) ;

$\forall F \forall G \forall H ((\lambda F) = ((\lambda G) + (\lambda H))$
 $\& \forall i (\leq[1, i] \& \leq[i, (\lambda G)] \Rightarrow (F' i) = (G' i))$
 $\& \forall i (\leq[(\lambda G), i] \& \leq[i, (\lambda F)] \Rightarrow (F' i) = (H' (i - (\lambda G))))$
 $\Rightarrow F \equiv (G \wedge H))$
! 63 (\forall I: 1, 62) ;

□

! 22. ;

$\vdash \forall F \forall G \forall H ((\lambda F) = ((\lambda G) + (\lambda H))$
 $\& \forall i (\leq[1, i] \& \leq[i, (\lambda G)] \Rightarrow (F' i) = (G' i))$
 $\& \forall i (\leq[1, i] \& \leq[i, (\lambda H)] \Rightarrow (F' ((\lambda G) + i)) = (H' i))$
 $\Rightarrow F \equiv (G \wedge H))$;

F, G, H , ! 1 (Prem) ;

$(\lambda F) = ((\lambda G) + (\lambda H))$
 $\& \forall i (\leq[1, i] \& \leq[i, (\lambda G)] \Rightarrow (F' i) = (G' i))$
 $\& \forall i (\leq[1, i] \& \leq[i, (\lambda H)] \Rightarrow (F' ((\lambda G) + i)) = (H' i))$
, ! 2 (Prem) ;

$(\lambda F) = ((\lambda G) + (\lambda H))$
 $\& \forall i (\leq[1, i] \& \leq[i, (\lambda G)] \Rightarrow (F' i) = (G' i))$
, ! 3 ($\&$ E: 2) ;

$(\lambda F) = ((\lambda G) + (\lambda H))$, ! 4 ($\&$ E: 2) ;

$\forall i (\leq[1, i] \& \leq[i, (\lambda H)] \Rightarrow (F' ((\lambda G) + i)) = (H' i))$
, ! 5 ($\&$ E: 2) ;

$\omega[(\lambda G)] \& \omega[(\lambda H)]$, ! 6 (\mathbb{T} E: v1.7, 4) ;

$\omega[(\lambda G)]$, ! 7 ($\&$ E: 6) ;

$\omega[(\lambda H)]$, ! 8 ($\&$ E: 6) ;

θG , ! 9 (\mathbb{T} E: c1.17, 7) ;

θH , ! 10 (\mathbb{T} E: c1.17, 8) ;

i	,!	11 (Prem)	i
$\langle [(\lambda G), i] \ \& \ \leq [i, (\lambda F)]$,!	12 (Prem)	i
$\langle [(\lambda G), i]$,!	13 (&E: 12)	i
$\leq [i, (\lambda F)]$,!	14 (&E: 12)	i
$(\langle [(\lambda G), i] \Rightarrow \leq [((\lambda G)+1), i])$,!	15 (\forall E: V4.36; (λG): C1.17,9)	i
$\langle [(\lambda G), i] \Rightarrow \leq [((\lambda G)+1), i]$,!	16 ($($)E: 15)	i
$\leq [((\lambda G)+1), i]$,!	17 (\Rightarrow E: 13,16)	i
$(\leq [((\lambda G)+1), i] \Rightarrow \leq [1, (i-(\lambda G))])$,!	18 (\forall E: V6.84; (λG): C1.17,9)	i
$\leq [((\lambda G)+1), i] \Rightarrow \leq [1, (i-(\lambda G))]$,!	19 ($($)E: 18)	i
$\leq [1, (i-(\lambda G))]$,!	20 (\Rightarrow E: 17,19)	i
$\leq [i, ((\lambda G)+(\lambda H))]$,!	21 (=E: 4,14)	i
$\leq [(\lambda G), i]$,!	22 (\mathbb{T} E: V5.7,20)	i
$\leq [i, ((\lambda G)+(\lambda H))] \ \& \ \leq [(\lambda G), i]$,!	23 (&I: 21,22)	i
$(\leq [i, ((\lambda G)+(\lambda H))] \ \& \ \leq [(\lambda G), i] \Rightarrow \leq [(i-(\lambda G)), (\lambda H)])$,!	24 (\forall E: V6.83; (λG): C1.17,9; (λH): C1.17,10)	i
$\leq [i, ((\lambda G)+(\lambda H))] \ \& \ \leq [(\lambda G), i] \Rightarrow \leq [(i-(\lambda G)), (\lambda H)]$,!	25 ($($)E: 24)	i
$\leq [(i-(\lambda G)), (\lambda H)]$,!	26 (\Rightarrow E: 23,25)	i
$\leq [1, (i-(\lambda G))] \ \& \ \leq [(i-(\lambda G)), (\lambda H)]$,!	27 (&I: 20,26)	i
$(\leq [1, (i-(\lambda G))] \ \& \ \leq [(i-(\lambda G)), (\lambda H)]$ $\Rightarrow (F'((\lambda G)+(i-(\lambda G)))) = (H'(i-(\lambda G))))$,!	28 (\forall E: 5; ($i-(\lambda G)$): V5.7,22)	i
$\leq [1, (i-(\lambda G))] \ \& \ \leq [(i-(\lambda G)), (\lambda H)]$ $\Rightarrow (F'((\lambda G)+(i-(\lambda G)))) = (H'(i-(\lambda G)))$,!	29 ($($)E: 28)	i
$(F'((\lambda G)+(i-(\lambda G)))) = (H'(i-(\lambda G)))$,!	30 (\Rightarrow E: 27,29)	i
$(\leq [(\lambda G), i] \Rightarrow ((\lambda G)+(i-(\lambda G))) = i)$			

,! 31 ($\forall E$: V6.4;
(λG): C1.17,9) i

$\leq[(\lambda G), i] \Rightarrow ((\lambda G)+(i -(\lambda G))) = i$,! 32 ($()E$: 31) i

$((\lambda G)+(i -(\lambda G))) = i$,! 33 ($\Rightarrow E$: 22,32) i

$(F' i) = (H'(i-(\lambda G)))$,! 34 ($=E$: 30,33) i

$\langle[(\lambda G), i] \ \& \ \leq[i, (\lambda F)] \Rightarrow (F' i) = (H'(i-(\lambda G)))$
,! 35 ($\Rightarrow I$: 12,34) i

$(\langle[(\lambda G), i] \ \& \ \leq[i, (\lambda F)] \Rightarrow (F' i) = (H'(i-(\lambda G))))$
,! 36 ($()I$: 35) i

$\forall i (\langle[(\lambda G), i] \ \& \ \leq[i, (\lambda F)] \Rightarrow (F' i) = (H'(i-(\lambda G))))$
,! 37 ($()I$: 36) i

$(\lambda F) = ((\lambda G)+(\lambda H))$
& $\forall i (\leq[1, i] \ \& \ \leq[i, (\lambda G)] \Rightarrow (F' i) = (G' i))$
& $\forall i (\langle[(\lambda G), i] \ \& \ \leq[i, (\lambda F)] \Rightarrow (F' i) = (H'(i-(\lambda G))))$
,! 38 ($\&I$: 3,37) i

$((\lambda F) = ((\lambda G)+(\lambda H))$
& $\forall i (\leq[1, i] \ \& \ \leq[i, (\lambda G)] \Rightarrow (F' i) = (G' i))$
& $\forall i (\langle[(\lambda G), i] \ \& \ \leq[i, (\lambda F)] \Rightarrow (F' i) = (H'(i-(\lambda G)))))$
 $\Rightarrow F \equiv (G^{\wedge}H))$
,! 39 ($\forall E$: P21) i

$(\lambda F) = ((\lambda G)+(\lambda H))$
& $\forall i (\leq[1, i] \ \& \ \leq[i, (\lambda G)] \Rightarrow (F' i) = (G' i))$
& $\forall i (\langle[(\lambda G), i] \ \& \ \leq[i, (\lambda F)] \Rightarrow (F' i) = (H'(i-(\lambda G))))$
 $\Rightarrow F \equiv (G^{\wedge}H)$
,! 40 ($()E$: 39) i

$F \equiv (G^{\wedge}H)$,! 41 ($\Rightarrow E$: 38,40) i

$(\lambda F) = ((\lambda G)+(\lambda H))$
& $\forall i (\leq[1, i] \ \& \ \leq[i, (\lambda G)] \Rightarrow (F' i) = (G' i))$
& $\forall i (\leq[1, i] \ \& \ \leq[i, (\lambda H)] \Rightarrow (F'((\lambda G)+i)) = (H' i))$
 $\Rightarrow F \equiv (G^{\wedge}H)$
,! 42 ($\Rightarrow I$: 2,41) i

$((\lambda F) = ((\lambda G)+(\lambda H))$
& $\forall i (\leq[1, i] \ \& \ \leq[i, (\lambda G)] \Rightarrow (F' i) = (G' i))$
& $\forall i (\leq[1, i] \ \& \ \leq[i, (\lambda H)] \Rightarrow (F'((\lambda G)+i)) = (H' i))$
 $\Rightarrow F \equiv (G^{\wedge}H))$
,! 43 ($()I$: 42) i

$\forall F \forall G \forall H ((\lambda F) = ((\lambda G)+(\lambda H))$
& $\forall i (\leq[1, i] \ \& \ \leq[i, (\lambda G)] \Rightarrow (F' i) = (G' i))$

$\& \forall i (\leq[1,i] \& \leq[i,(\lambda H)] \Rightarrow (F'((\lambda G)+i)) = (H'i))$
 $\Rightarrow F \equiv (G^H))$

! 44 ($\forall I$: 1,43) i

□

! 23.

$\vdash \forall F \forall a (\theta F \Rightarrow (((1 \blacksquare a)^F)'1) = a)$

F, a

,! 1 (Prem) i

θF

,! 2 (Prem) i

$\theta F \& \leq[1,1]$

,! 3 ($\&I$: V3.49,2) i

$(\lambda(1 \blacksquare a)) = 1$

,! 4 ($\forall E$: C1.48) i

$\leq[1,(\lambda(1 \blacksquare a))]$

,! 5 ($=E$: V3.49,4) i

$\theta F \& \leq[1,1] \& \leq[1,(\lambda(1 \blacksquare a))]$

,! 6 ($\&I$: 3,5) i

$(\theta F \& \leq[1,1] \& \leq[1,(\lambda(1 \blacksquare a))])$

$\Rightarrow (((1 \blacksquare a)^F)'1) = ((1 \blacksquare a)'1))$

,! 7 ($\forall E$: P16) i

$\theta F \& \leq[1,1] \& \leq[1,(\lambda(1 \blacksquare a))]$

$\Rightarrow (((1 \blacksquare a)^F)'1) = ((1 \blacksquare a)'1)$

,! 8 ($()E$: 7) i

$((1 \blacksquare a)^F)'1) = ((1 \blacksquare a)'1)$

,! 9 ($\Rightarrow E$: 6,8) i

$((1 \blacksquare a)'1) = a$

,! 10 ($\forall E$: IIII12.24) i

$((1 \blacksquare a)^F)'1) = a$

,! 11 ($=E$: 9,10) i

$\theta F \Rightarrow (((1 \blacksquare a)^F)'1) = a$

,! 12 ($\Rightarrow I$: 2,11) i

$(\theta F \Rightarrow (((1 \blacksquare a)^F)'1) = a)$

,! 13 ($()I$: 12) i

$\forall F \forall a (\theta F \Rightarrow (((1 \blacksquare a)^F)'1) = a)$

! 14 ($\forall I$: 1,13) i

□

! 24.

$\vdash \forall F \forall a \forall i (\leq[1,i] \& \leq[i,(\lambda F)]$

$\Rightarrow (((1 \blacksquare a)^F)'(i+1)) = (F'i))$

F, a, i

,! 1 (Prem) i

$\leq[1,i] \& \leq[i,(\lambda F)]$

,! 2 (Prem) i

$\leq[1,i]$

,! 3 ($\&E$: 2) i

$\theta(1 \multimap a)$,! 4 ($\forall E$: C1.14) ;
 $\theta(1 \multimap a) \ \& \ \leq[1, i] \ \& \ \leq[i, (\lambda F)]$,! 5 ($\&I$: 2,4) ;
 $(\theta(1 \multimap a) \ \& \ \leq[1, i] \ \& \ \leq[i, (\lambda F)])$
 $\Rightarrow (((1 \multimap a)^F)^\wedge((\lambda(1 \multimap a)) + i)) = (F'i)$,! 6 ($\forall E$: P19) ;
 $\theta(1 \multimap a) \ \& \ \leq[1, i] \ \& \ \leq[i, (\lambda F)]$
 $\Rightarrow (((1 \multimap a)^F)^\wedge((\lambda(1 \multimap a)) + i)) = (F'i)$,! 7 ($()E$: 6) ;
 $((((1 \multimap a)^F)^\wedge((\lambda(1 \multimap a)) + i)) = (F'i))$,! 8 ($\Rightarrow E$: 5,7) ;
 $(\lambda(1 \multimap a)) = 1$,! 9 ($\forall E$: C1.48) ;
 $((((1 \multimap a)^F)^\wedge(1 + i)) = (F'i))$,! 10 ($=E$: 8,9) ;
 $(\leq[1, i] \Rightarrow \omega[1] \ \& \ \omega[i])$,! 11 ($\forall E$: V3.5) ;
 $\leq[1, i] \Rightarrow \omega[1] \ \& \ \omega[i]$,! 12 ($()E$: 11) ;
 $\omega[1] \ \& \ \omega[i]$,! 13 ($\Rightarrow E$: 3,12) ;
 $(\omega[1] \ \& \ \omega[i] \Rightarrow (1+i) = (i+1))$,! 14 ($\forall E$: V2.5) ;
 $\omega[1] \ \& \ \omega[i] \Rightarrow (1+i) = (i+1)$,! 15 ($()E$: 14) ;
 $(1+i) = (i+1)$,! 16 ($\Rightarrow E$: 13,15) ;
 $((((1 \multimap a)^F)^\wedge(i+1)) = (F'i))$,! 17 ($=E$: 10,16) ;
 $\leq[1, i] \ \& \ \leq[i, (\lambda F)] \Rightarrow (((1 \multimap a)^F)^\wedge(i+1)) = (F'i)$,! 18 ($\Rightarrow I$: 2,17) ;
 $(\leq[1, i] \ \& \ \leq[i, (\lambda F)] \Rightarrow (((1 \multimap a)^F)^\wedge(i+1)) = (F'i))$,! 19 ($()I$: 18) ;
 $\forall F \forall a \forall i (\leq[1, i] \ \& \ \leq[i, (\lambda F)])$
 $\Rightarrow (((1 \multimap a)^F)^\wedge(i+1)) = (F'i)$! 20 ($\forall I$: 1,19) ;
 \square
! 25. ;
 $\vdash \forall F \forall G \forall a \forall i (\leq[1, i] \ \& \ \leq[i, (\lambda F)] \ \& \ G \equiv ((1 \multimap a)^F)$
 $\Rightarrow (G'(i+1)) = (F'i)$;
F, G, a, i ,! 1 (Prem) ;
 $\leq[1, i] \ \& \ \leq[i, (\lambda F)] \ \& \ G \equiv ((1 \multimap a)^F)$,! 2 (Prem) ;

$\leq[1, i] \ \& \ \leq[i, (\lambda F)]$,! 3 (&E: 2) i
 $G \equiv ((1 \ \blacksquare \ a)^F)$,! 4 (&E: 2) i
 $(\leq[1, i] \ \& \ \leq[i, (\lambda F)] \Rightarrow (((1 \ \blacksquare \ a)^F)^{(i+1)}) = (F'i))$
,! 5 (\forall E: P24) i
 $\leq[1, i] \ \& \ \leq[i, (\lambda F)] \Rightarrow (((1 \ \blacksquare \ a)^F)^{(i+1)}) = (F'i)$
,! 6(\Rightarrow E: 5) i
 $((1 \ \blacksquare \ a)^F)^{(i+1)} = (F'i)$,! 7 (\Rightarrow E: 3,6) i
 $f((1 \ \blacksquare \ a)^F) \ \& \ (((1 \ \blacksquare \ a)^F)^D)[(i+1)]$,! 8 (\mathbb{T} E: III8.20) i
 $f((1 \ \blacksquare \ a)^F) \ \& \ (((1 \ \blacksquare \ a)^F)^D)[(i+1)] \ \& \ G \equiv ((1 \ \blacksquare \ a)^F)$
,! 9 (&I: 4,8) i
 $((1 \ \blacksquare \ a)^F)^D[(i+1)]$,! 10 (&E: 8) i
 $\omega[i] \ \& \ \omega[1]$,! 11 (\mathbb{T} E: V1.7,10) i
 $(f((1 \ \blacksquare \ a)^F) \ \& \ (((1 \ \blacksquare \ a)^F)^D)[(i+1)] \ \& \ G \equiv ((1 \ \blacksquare \ a)^F)$
 $\Rightarrow (((1 \ \blacksquare \ a)^F)^{(i+1)}) = (G'(i+1))$
,! 12 (\forall E: III8.26;
 $(i+1)$: V1.7,11) i
 $f((1 \ \blacksquare \ a)^F) \ \& \ (((1 \ \blacksquare \ a)^F)^D)[(i+1)] \ \& \ G \equiv ((1 \ \blacksquare \ a)^F)$
 $\Rightarrow (((1 \ \blacksquare \ a)^F)^{(i+1)}) = (G'(i+1))$
,! 13 (\Rightarrow E: 12) i
 $((1 \ \blacksquare \ a)^F)^{(i+1)} = (G'(i+1))$,! 14 (\Rightarrow E: 9,13) i
 $(G'(i+1)) = (F'i)$,! 15 (=E: 7,14) i
 $\leq[1, i] \ \& \ \leq[i, (\lambda F)] \ \& \ G \equiv ((1 \ \blacksquare \ a)^F) \Rightarrow (G'(i+1)) = (F'i)$
,! 16 (\Rightarrow I: 2,15) i
 $(\leq[1, i] \ \& \ \leq[i, (\lambda F)] \ \& \ G \equiv ((1 \ \blacksquare \ a)^F) \Rightarrow (G'(i+1)) = (F'i))$
,! 17 (\Rightarrow I: 16) i
 $\forall F \forall G \forall a \forall i (\leq[1, i] \ \& \ \leq[i, (\lambda F)] \ \& \ G \equiv ((1 \ \blacksquare \ a)^F)$
 $\Rightarrow (G'(i+1)) = (F'i))$
! 18 (\forall I: 1,17) i
□
! 26. i
 $\vdash \forall F \forall a \forall i (\theta F \ \& \ (F'i) = a \Rightarrow (((1 \ \blacksquare \ a)^F)^{(i+1)}) = a)$ i
 F, a, i ,! 1 (Prem) i
 $\theta F \ \& \ (F'i) = a$,! 2 (Prem) i

$(\theta F \ \& \ (F' i) = a \Rightarrow \leq[1, i] \ \& \ \leq[i, (\lambda F)])$
, ! 3 ($\forall E$: C1.29) i

$\theta F \ \& \ (F' i) = a \Rightarrow \leq[1, i] \ \& \ \leq[i, (\lambda F)]$, ! 4 ($()E$: 3) i

$\leq[1, i] \ \& \ \leq[i, (\lambda F)]$, ! 5 ($\Rightarrow E$: 2,4) i

$(\leq[1, i] \ \& \ \leq[i, (\lambda F)] \Rightarrow (((1 \ \blacksquare \ a)^F)'(i+1)) = (F' i))$
, ! 6 ($\forall E$: P24) i

$\leq[1, i] \ \& \ \leq[i, (\lambda F)] \Rightarrow (((1 \ \blacksquare \ a)^F)'(i+1)) = (F' i)$
, ! 7 ($()E$: 6) i

$(((1 \ \blacksquare \ a)^F)'(i+1)) = (F' i)$, ! 8 ($\Rightarrow E$: 5,7) i

$(F' i) = a$, ! 9 ($\&E$: 2) i

$(((1 \ \blacksquare \ a)^F)'(i+1)) = a$, ! 10 ($=E$: 8,9) i

$\theta F \ \& \ (F' i) = a \Rightarrow (((1 \ \blacksquare \ a)^F)'(i+1)) = (F' i)$
, ! 11 ($\Rightarrow I$: 2,10) i

$(\theta F \ \& \ (F' i) = a \Rightarrow (((1 \ \blacksquare \ a)^F)'(i+1)) = (F' i))$
, ! 12 ($()I$: 11) i

$\forall F \forall a \forall i (\theta F \ \& \ (F' i) = a \Rightarrow (((1 \ \blacksquare \ a)^F)'(i+1)) = a)$
! 13 ($\forall I$: 1,12) i

□

! 27. i

$\vdash \forall F \forall G \forall a \forall b \forall i (\theta F \ \& \ G \equiv ((1 \ \blacksquare \ a)^F) \ \& \ (G'(i+1)) = b \ \& \ \leq[1, i]$
 $\Rightarrow (F' i) = b)$ i

F, G, a, b, i , ! 1 (Prem) i

$\theta F \ \& \ G \equiv ((1 \ \blacksquare \ a)^F) \ \& \ (G'(i+1)) = b \ \& \ \leq[1, i]$
, ! 2 (Prem) i

θF , ! 3 ($\&E$: 2) i

$G \equiv ((1 \ \blacksquare \ a)^F) \ \& \ (G'(i+1)) = b$, ! 4 ($\&E$: 2) i

$\leq[1, i]$, ! 5 ($\&E$: 2) i

$(\leq[1, i] \Rightarrow \omega[i])$, ! 6 ($\forall E$: V3.7) i

$\leq[1, i] \Rightarrow \omega[i]$, ! 7 ($()E$: 6) i

$\omega[i]$, ! 8 ($\Rightarrow E$: 5,7) i

$\omega[i] \ \& \ \omega[1]$, ! 9 ($\&I$: IV9.2,8) i

$(G \equiv ((1 \ \blacksquare \ a)^F) \ \& \ (G'(i+1)) = b$

$\Rightarrow ((1 \dashv a)^F)^{(i+1)} = b$

,! 10 ($\forall E$: III8.27;
 $(i+1)$: V1.7,9) i

$G \equiv ((1 \dashv a)^F) \ \& \ (G^{(i+1)}) = b$
 $\Rightarrow ((1 \dashv a)^F)^{(i+1)} = b$

,! 11 ($()E$: 10) i

$((1 \dashv a)^F)^{(i+1)} = b$

,! 12 ($\Rightarrow E$: 4,11) i

$\theta(1 \dashv a)$

,! 13 ($\forall E$: C1.14) i

$\theta(1 \dashv a) \ \& \ \theta F$

,! 14 ($\&I$: 3,13) i

$(\theta(1 \dashv a) \ \& \ \theta F \Rightarrow \theta((1 \dashv a)^F))$

,! 15 ($\forall E$: P10) i

$\theta(1 \dashv a) \ \& \ \theta F \Rightarrow \theta((1 \dashv a)^F)$

,! 16 ($()E$: 15) i

$\theta((1 \dashv a)^F)$

,! 17 ($\Rightarrow E$: 14,16) i

$\theta((1 \dashv a)^F) \ \& \ (((1 \dashv a)^F)^{(i+1)}) = b$

,! 18 ($\&I$: 12,17) i

$(\theta((1 \dashv a)^F) \ \& \ (((1 \dashv a)^F)^{(i+1)}) = b$
 $\Rightarrow \leq[1, (i+1)] \ \& \ \leq[(i+1), (\lambda((1 \dashv a)^F))]$

,! 19 ($\forall E$: C1.29;
 $(i+1)$: V1.7,9) i

$\theta((1 \dashv a)^F) \ \& \ (((1 \dashv a)^F)^{(i+1)}) = b$
 $\Rightarrow \leq[1, (i+1)] \ \& \ \leq[(i+1), (\lambda((1 \dashv a)^F))]$

,! 20 ($()E$: 19) i

$\leq[1, (i+1)] \ \& \ \leq[(i+1), (\lambda((1 \dashv a)^F))]$

,! 21 ($\Rightarrow E$: 18,20) i

$\leq[(i+1), (\lambda((1 \dashv a)^F))]$

,! 22 ($\&E$: 21) i

$(\omega[i] \ \& \ \omega[1] \Rightarrow (i+1) = (1+i))$

,! 23 ($\forall E$: V2.5) i

$\omega[i] \ \& \ \omega[1] \Rightarrow (i+1) = (1+i)$

,! 24 ($()E$: 23) i

$(i+1) = (1+i)$

,! 25 ($\Rightarrow E$: 9,24) i

$\leq[(1+i), (\lambda((1 \dashv a)^F))]$

,! 26 ($=E$: 22,25) i

$(\theta(1 \dashv a) \ \& \ \theta F \Rightarrow (\lambda((1 \dashv a)^F)) = ((\lambda(1 \dashv a)) + (\lambda F)))$

,! 27 ($\forall E$: P11) i

$\theta(1 \dashv a) \ \& \ \theta F \Rightarrow (\lambda((1 \dashv a)^F)) = ((\lambda(1 \dashv a)) + (\lambda F))$

,! 28 ($()E$: 27) i

$(\lambda((1 \dashv a)^F)) = ((\lambda(1 \dashv a)) + (\lambda F))$

,! 29 ($\Rightarrow E$: 14,28) i

$(\lambda(1 \dashv a)) = 1$

,! 30 ($\forall E$: C1.48) i

$(\theta(1 \cdot a) \ \& \ \theta F \Rightarrow (\lambda((1 \cdot a)^F)) = ((\lambda(1 \cdot a)) + (\lambda F)))$,! 7 ($\forall E$: P11) ;
 $\theta(1 \cdot a) \ \& \ \theta F \Rightarrow (\lambda((1 \cdot a)^F)) = ((\lambda(1 \cdot a)) + (\lambda F))$,! 8 ($()E$: 7) ;
 $(\lambda((1 \cdot a)^F)) = ((\lambda(1 \cdot a)) + (\lambda F))$,! 9 ($\Rightarrow E$: 5,7) ;
 $(\lambda((1 \cdot a)^F)) = ((\lambda(1 \cdot a)) + n)$,! 10 ($=E$: 2,9) ;
 $(\lambda(1 \cdot a)) = 1$,! 11 ($\forall E$: C1.48) ;
 $(\lambda((1 \cdot a)^F)) = (1+n)$,! 12 ($=E$: 10,11) ;
 $\omega[1] \ \& \ \omega[n]$,! 13 ($\mathbb{T}E$: V1.7,12) ;
 $(\omega[1] \ \& \ \omega[n] \Rightarrow (1+n) = (n+1))$,! 14 ($\forall E$: V2.5) ;
 $\omega[1] \ \& \ \omega[n] \Rightarrow (1+n) = (n+1)$,! 15 ($()E$: 14) ;
 $(1+n) = (n+1)$,! 16 ($\Rightarrow E$: 13,15) ;
 $(\lambda((1 \cdot a)^F)) = (n+1)$,! 17 ($=E$: 12,16) ;
 $(\theta F \Rightarrow (((1 \cdot a)^F)'1) = a)$,! 18 ($\forall E$: P23) ;
 $\theta F \Rightarrow (((1 \cdot a)^F)'1) = a$,! 19 ($()E$: 18) ;
 $(((1 \cdot a)^F)'1) = a$,! 20 ($\Rightarrow E$: 3,19) ;
 $(\lambda((1 \cdot a)^F)) = (n+1) \ \& \ (((1 \cdot a)^F)'1) = a$,! 21 ($\&I$: 17,20) ;
i ,! 22 (Prem) ;
 $\leq[1,i] \ \& \ \leq[i,n]$,! 23 (Prem) ;
 $\leq[1,i] \ \& \ \leq[i,(\lambda F)]$,! 24 ($=E$: 2,23) ;
 $(\leq[1,i] \ \& \ \leq[i,(\lambda F)] \Rightarrow (((1 \cdot a)^F)'(i+1)) = (F'i))$,! 25 ($\forall E$: P24) ;
 $\leq[1,i] \ \& \ \leq[i,(\lambda F)] \Rightarrow (((1 \cdot a)^F)'(i+1)) = (F'i)$,! 26 ($()E$: 25) ;
 $((((1 \cdot a)^F)'(i+1))) = (F'i)$,! 27 ($\Rightarrow E$: 24,26) ;
 $\leq[1,i] \ \& \ \leq[i,n] \Rightarrow (((1 \cdot a)^F)'(i+1)) = (F'i)$,! 28 ($\Rightarrow I$: 23,27) ;
 $(\leq[1,i] \ \& \ \leq[i,n] \Rightarrow (((1 \cdot a)^F)'(i+1)) = (F'i))$,! 29 ($()I$: 28) ;
 $\forall i (\leq[1,i] \ \& \ \leq[i,n] \Rightarrow (((1 \cdot a)^F)'(i+1)) = (F'i))$

,! 30 ($\forall I: 1,29$) i

$(\lambda((1 \ \square \ a)^F)) = (n+1) \ \& \ (((1 \ \square \ a)^F)'1) = a$
 $\& \ \forall i \ (\leq[1,i] \ \& \ \leq[i,n] \Rightarrow \ (((1 \ \square \ a)^F)'(i+1)) = (F'i))$
,! 31 ($\&I: 21,30$) i

$(\lambda F) = n$
 $\Rightarrow (\lambda((1 \ \square \ a)^F)) = (n+1)$
 $\ \& \ (((1 \ \square \ a)^F)'1) = a$
 $\ \& \ \forall i \ (\leq[1,i] \ \& \ \leq[i,n] \Rightarrow \ (((1 \ \square \ a)^F)'(i+1)) = (F'i))$
,! 32 ($\Rightarrow I: 2,31$) i

$(\ (\lambda F) = n$
 $\Rightarrow (\lambda((1 \ \square \ a)^F)) = (n+1)$
 $\ \& \ (((1 \ \square \ a)^F)'1) = a$
 $\ \& \ \forall i \ (\leq[1,i] \ \& \ \leq[i,n] \Rightarrow \ (((1 \ \square \ a)^F)'(i+1)) = (F'i)))$
,! 33 ($()I: 32$) i

$\forall F \forall a \forall n \ (\ (\lambda F) = n$
 $\Rightarrow (\lambda((1 \ \square \ a)^F)) = (n+1)$
 $\ \& \ (((1 \ \square \ a)^F)'1) = a$
 $\ \& \ \forall i \ (\leq[1,i] \ \& \ \leq[i,n] \Rightarrow \ (((1 \ \square \ a)^F)'(i+1)) = (F'i)))$
! 34 ($\forall I: 1,33$) i

□

! 29. i

$\vdash \forall F \forall a \forall n \ (\ (\lambda F) = n$
 $\Rightarrow (\lambda(F^{\wedge}(1 \ \square \ a))) = (n+1)$
 $\ \& \ ((F^{\wedge}(1 \ \square \ a))'^{(n+1)}) = a)$
i

F, a, n ,! 1 (Prem) i

$(\lambda F) = n$,! 2 (Prem) i

θF ,! 3 ($\mathbb{T}E: C1.17,2$) i

$\theta (1 \ \square \ a)$,! 4 ($\forall E: C1.14$) i

$\theta F \ \& \ \theta (1 \ \square \ a)$,! 5 ($\&I: 3,4$) i

$(\ \theta F \ \& \ \theta (1 \ \square \ a) \Rightarrow (\lambda(F^{\wedge}(1 \ \square \ a))) = ((\lambda F) + (\lambda(1 \ \square \ a))))$
,! 6 ($\forall E: P11$) i

$\theta F \ \& \ \theta (1 \ \square \ a) \Rightarrow (\lambda(F^{\wedge}(1 \ \square \ a))) = ((\lambda F) + (\lambda(1 \ \square \ a)))$
,! 7 ($()E: 6$) i

$(\lambda(F^{\wedge}(1 \ \square \ a))) = ((\lambda F) + (\lambda(1 \ \square \ a)))$,! 8 ($\Rightarrow E: 5,7$) i

$(\lambda(F^{\wedge}(1 \ \square \ a))) = (n + (\lambda(1 \ \square \ a)))$,! 9 ($=E: 2,8$) i

$(\lambda(1 \dashv a)) = 1$,! 10 ($\forall E$: C1.48) ;
 $(\lambda(F^{\wedge}(1 \dashv a))) = (n+1)$,! 11 ($=E$: 9,10) ;
 $\theta F \ \& \ \leq[1,1]$,! 12 ($\&I$: V3.49,3) ;
 $\leq[1,(\lambda(1 \dashv a))]$,! 13 ($=E$: V3.49,10) ;
 $\theta F \ \& \ \leq[1,1] \ \& \ \leq[1,(\lambda(1 \dashv a))]$,! 14 ($\&I$: 12,13) ;
 $(\theta F \ \& \ \leq[1,1] \ \& \ \leq[1,(\lambda(1 \dashv a))])$
 $\Rightarrow ((F^{\wedge}(1 \dashv a))^{\wedge}((\lambda F)+1)) = ((1 \dashv a)^{\wedge}1)$,! 15 ($\forall E$: P19) ;
 $\theta F \ \& \ \leq[1,1] \ \& \ \leq[1,(\lambda(1 \dashv a))]$
 $\Rightarrow ((F^{\wedge}(1 \dashv a))^{\wedge}((\lambda F)+1)) = ((1 \dashv a)^{\wedge}1)$,! 16 ($()E$: 15) ;
 $((F^{\wedge}(1 \dashv a))^{\wedge}((\lambda F)+1)) = ((1 \dashv a)^{\wedge}1)$,! 17 ($\Rightarrow E$: 14,16) ;
 $((F^{\wedge}(1 \dashv a))^{\wedge}(n+1)) = ((1 \dashv a)^{\wedge}1)$,! 18 ($=E$: 2,17) ;
 $((1 \dashv a)^{\wedge}1) = a$,! 19 ($\forall E$: IIII2.24) ;
 $((F^{\wedge}(1 \dashv a))^{\wedge}(n+1)) = a$,! 20 ($=E$: 18,19) ;
 $(\lambda(F^{\wedge}(1 \dashv a))) = (n+1) \ \& \ ((F^{\wedge}(1 \dashv a))^{\wedge}(n+1)) = a$,! 21 ($\&I$: 11,20) ;
 $(\lambda F) = n \Rightarrow (\lambda(F^{\wedge}(1 \dashv a))) = (n+1) \ \& \ ((F^{\wedge}(1 \dashv a))^{\wedge}(n+1)) = a$,! 22 ($\Rightarrow I$: 2,21) ;
 $(\lambda F) = n$
 $\Rightarrow (\lambda(F^{\wedge}(1 \dashv a))) = (n+1) \ \& \ ((F^{\wedge}(1 \dashv a))^{\wedge}(n+1)) = a$,! 23 ($()I$: 22) ;
 $\forall F \forall a \forall n \ ((\lambda F) = n$
 $\Rightarrow (\lambda(F^{\wedge}(1 \dashv a))) = (n+1) \ \& \ ((F^{\wedge}(1 \dashv a))^{\wedge}(n+1)) = a)$! 24 ($\forall I$: 1,23) ;
 \square
! 30. ;
 $\vdash \forall F \ (\theta F \Rightarrow (F^{\wedge}\Phi) \equiv F)$;
F ,! 1 (Prem) ;
 θF ,! 2 (Prem) ;
 $((\Theta(\lambda F)) \circ \Phi) \equiv \Phi$,! 3 ($\forall E$: IIII10.13) ;

$(((\Theta(\lambda F)) \circ \Phi) \equiv \Phi \Rightarrow (F \sqcup ((\Theta(\lambda F)) \circ \Phi)) \equiv F)$,! 4 ($\forall E$: III4.12) ;
$((\Theta(\lambda F)) \circ \Phi) \equiv \Phi \Rightarrow (F \sqcup ((\Theta(\lambda F)) \circ \Phi)) \equiv F$,! 5 ($()E$: 4) ;
$(F \sqcup ((\Theta(\lambda F)) \circ \Phi)) \equiv F$,! 6 ($\Rightarrow E$: 3,5) ;
$(F \wedge \Phi) \equiv F$,! 7 ($\mathbb{D}I$: P1,6) ;
$\theta F \Rightarrow (F \wedge \Phi) \equiv F$,! 8 ($\Rightarrow I$: 2,7) ;
$(\theta F \Rightarrow (F \wedge \Phi)) \equiv F$,! 9 ($()I$: 8) ;
$\forall F (\theta F \Rightarrow (F \wedge \Phi)) \equiv F$! 10 ($\forall I$: 1,9) ;
\square	
! 31.	; i
$\vdash \forall F (\theta F \Rightarrow (\Phi \wedge F) \equiv F)$; i
F	,! 1 (Prem) ;
θF	,! 2 (Prem) ;
$(\theta F \Rightarrow (F^D) \subseteq \omega)$,! 3 ($\forall E$: C1.5) ;
$\theta F \Rightarrow (F^D) \subseteq \omega$,! 4 ($()E$: 3) ;
$(F^D) \subseteq \omega$,! 5 ($\Rightarrow E$: 2,4) ;
$((F^D) \subseteq \omega \Rightarrow ((I\omega) \circ F) \equiv F)$,! 6 ($\forall E$: III11.28) ;
$(F^D) \subseteq \omega \Rightarrow ((I\omega) \circ F) \equiv F$,! 7 ($()E$: 6) ;
$((I\omega) \circ F) \equiv F$,! 8 ($\Rightarrow E$: 5,7) ;
$((\Theta 0) \equiv (I\omega) \Rightarrow ((\Theta 0) \circ F) \equiv ((I\omega) \circ F))$,! 9 ($\forall E$: III10.9) ;
$(\Theta 0) \equiv (I\omega) \Rightarrow ((\Theta 0) \circ F) \equiv ((I\omega) \circ F)$,! 10 ($()E$: 9) ;
$((\Theta 0) \circ F) \equiv ((I\omega) \circ F)$,! 11 ($\Rightarrow E$: V10.25,10) ;
$((\Theta 0) \circ F) \equiv ((I\omega) \circ F) \& ((I\omega) \circ F) \equiv F$,! 12 ($\&I$: 8,11) ;
$((\Theta 0) \circ F) \equiv ((I\omega) \circ F) \& ((I\omega) \circ F) \equiv F$ $\Rightarrow ((\Theta 0) \circ F) \equiv F$,! 13 ($\forall E$: III1.15) ;

$((\Theta 0) \circ F) \equiv ((I\omega) \circ F) \ \& \ ((I\omega) \circ F) \equiv F \Rightarrow ((\Theta 0) \circ F) \equiv F$,! 14 ((E: 13)	i
$((\Theta 0) \circ F) \equiv F$,! 15 (\Rightarrow E: 12,14)	i
$(\Phi \sqcup ((\Theta 0) \circ F)) \equiv ((\Theta 0) \circ F)$,! 16 (\forall E: III4.11)	i
$(\Phi \sqcup ((\Theta 0) \circ F)) \equiv ((\Theta 0) \circ F) \ \& \ ((\Theta 0) \circ F) \equiv F$,! 17 ($\&$ I: 15,16)	i
$(\Phi \sqcup ((\Theta 0) \circ F)) \equiv ((\Theta 0) \circ F) \ \& \ ((\Theta 0) \circ F) \equiv F$ $\Rightarrow (\Phi \sqcup ((\Theta 0) \circ F)) \equiv F$,! 18 (\forall E: III1.15)	i
$(\Phi \sqcup ((\Theta 0) \circ F)) \equiv ((\Theta 0) \circ F) \ \& \ ((\Theta 0) \circ F) \equiv F$ $\Rightarrow (\Phi \sqcup ((\Theta 0) \circ F)) \equiv F$,! 19 ((E: 18)	i
$(\Phi \sqcup ((\Theta 0) \circ F)) \equiv F$,! 20 (\Rightarrow E: 17,19)	i
$(\Phi \sqcup ((\Theta(\lambda\Phi)) \circ F)) \equiv F$,! 21 (=E: C1.46,20)	i
$(\Phi \wedge F) \equiv F$,! 22 (\mathbb{D} I: P1,21)	i
$\theta F \Rightarrow (\Phi \wedge F) \equiv F$,! 23 (\Rightarrow I: 2,22)	i
$(\theta F \Rightarrow (\Phi \wedge F) \equiv F)$,! 24 ((I: 23)	i
$\forall F (\theta F \Rightarrow (\Phi \wedge F) \equiv F)$! 25 (\forall I: 1,24)	i
\square		
! 32.		i
$\vdash \forall F \forall G \forall H (\theta F \ \& \ \theta G \ \& \ \theta H \Rightarrow (F \wedge (G \wedge H)) \equiv ((F \wedge G) \wedge H))$		i
F, G, H	,! 1 (Prem)	i
$\theta F \ \& \ \theta G \ \& \ \theta H$,! 2 (Prem)	i
θF	,! 3 ($\&$ E: 2)	i
θG	,! 4 ($\&$ E: 2)	i
θH	,! 5 ($\&$ E: 2)	i
$\theta F \ \& \ \theta G$,! 6 ($\&$ I: 3,4)	i
$(\theta F \ \& \ \theta G \Rightarrow \theta (F \wedge G))$,! 7 (\forall E: P14)	i
$\theta F \ \& \ \theta G \Rightarrow \theta (F \wedge G)$,! 8 ((E: 7)	i

$\theta (F \wedge G)$,! 9 (\Rightarrow E: 6,8) i

$\theta G \ \& \ \theta H$,! 10 ($\&$ I: 4,5) i

$(\theta G \ \& \ \theta H \Rightarrow \theta (G \wedge H))$,! 11 (\forall E: P14) i

$\theta G \ \& \ \theta H \Rightarrow \theta (G \wedge H)$,! 12 ($(\)$ E: 11) i

$\theta (G \wedge H)$,! 13 (\Rightarrow E: 10,12) i

$\theta F \ \& \ \theta (G \wedge H)$,! 14 ($\&$ I: 3,13) i

$(\theta F \ \& \ \theta (G \wedge H) \Rightarrow \theta (F \wedge (G \wedge H)))$,! 15 (\forall E: P14) i

$\theta F \ \& \ \theta (G \wedge H) \Rightarrow \theta (F \wedge (G \wedge H))$,! 16 ($(\)$ E: 15) i

$\theta (F \wedge (G \wedge H))$,! 17 (\Rightarrow E: 14,16) i

$((\lambda(F \wedge (G \wedge H))) = ((\lambda(F \wedge G)) + (\lambda H))$
& $\forall i (\leq[1, i] \ \& \ \leq[i, (\lambda(F \wedge G))])$
 $\Rightarrow ((F \wedge (G \wedge H))'i) = ((F \wedge G)'i))$
& $\forall i (\leq[1, i] \ \& \ \leq[i, (\lambda H)])$
 $\Rightarrow ((F \wedge (G \wedge H))'((\lambda(F \wedge G)) + i)) = (H'i))$
 $\Rightarrow (F \wedge (G \wedge H)) \equiv ((F \wedge G) \wedge H))$
,! 18 (\forall E: P22) i

$(\lambda(F \wedge (G \wedge H))) = ((\lambda(F \wedge G)) + (\lambda H))$
& $\forall i (\leq[1, i] \ \& \ \leq[i, (\lambda(F \wedge G))]) \Rightarrow ((F \wedge (G \wedge H))'i) = ((F \wedge G)'i))$
& $\forall i (\leq[1, i] \ \& \ \leq[i, (\lambda H)])$
 $\Rightarrow ((F \wedge (G \wedge H))'((\lambda(F \wedge G)) + i)) = (H'i))$
 $\Rightarrow (F \wedge (G \wedge H)) \equiv ((F \wedge G) \wedge H)$
,! 19 ($(\)$ E: 18) i

! To prove: $(\lambda(F \wedge (G \wedge H))) = ((\lambda(F \wedge G)) + (\lambda H))$ i

$(\theta F \ \& \ \theta (G \wedge H) \Rightarrow (\lambda(F \wedge (G \wedge H))) = ((\lambda F) + (\lambda(G \wedge H))))$
,! 20 (\forall E: P11) i

$\theta F \ \& \ \theta (G \wedge H) \Rightarrow (\lambda(F \wedge (G \wedge H))) = ((\lambda F) + (\lambda(G \wedge H)))$
,! 21 ($(\)$ E: 20) i

$(\lambda(F \wedge (G \wedge H))) = ((\lambda F) + (\lambda(G \wedge H)))$,! 22 (\Rightarrow E: 14,21) i

$(\theta G \ \& \ \theta H \Rightarrow (\lambda(G \wedge H)) = ((\lambda G) + (\lambda H)))$
,! 23 (\forall E: P11) i

$\theta G \ \& \ \theta H \Rightarrow (\lambda(G \wedge H)) = ((\lambda G) + (\lambda H))$,! 24 ($(\)$ E: 23) i

$(\lambda(G \wedge H)) = ((\lambda G) + (\lambda H))$,! 25 (\Rightarrow E: 10,24) i

$(\lambda(F \wedge (G \wedge H))) = ((\lambda F) + ((\lambda G) + (\lambda H)))$,! 26 ($=$ E: 22,25) i

$$\begin{aligned}
& (\lambda(\mathbf{F}^{\wedge}(\mathbf{G}^{\wedge}\mathbf{H}))) = ((\lambda\mathbf{F}) + ((\lambda\mathbf{G}) + (\lambda\mathbf{H}))) \\
& \Rightarrow (\lambda(\mathbf{F}^{\wedge}(\mathbf{G}^{\wedge}\mathbf{H}))) = (((\lambda\mathbf{F}) + (\lambda\mathbf{G})) + (\lambda\mathbf{H}))) \\
& \qquad \qquad \qquad ,! 27 (\forall\text{E: v2.17;} \\
& \qquad \qquad \qquad (\lambda(\mathbf{F}^{\wedge}(\mathbf{G}^{\wedge}\mathbf{H}))) : \text{C1.17,17;} \\
& \qquad \qquad \qquad (\lambda\mathbf{F}) : \text{C1.17,3;} \\
& \qquad \qquad \qquad (\lambda\mathbf{G}) : \text{C1.17,4;} \\
& \qquad \qquad \qquad (\lambda\mathbf{H}) : \text{C1.17,5} \qquad \qquad \qquad ;
\end{aligned}$$

$$\begin{aligned}
& (\lambda(\mathbf{F}^{\wedge}(\mathbf{G}^{\wedge}\mathbf{H}))) = ((\lambda\mathbf{F}) + ((\lambda\mathbf{G}) + (\lambda\mathbf{H}))) \\
& \Rightarrow (\lambda(\mathbf{F}^{\wedge}(\mathbf{G}^{\wedge}\mathbf{H}))) = (((\lambda\mathbf{F}) + (\lambda\mathbf{G})) + (\lambda\mathbf{H})) \\
& \qquad \qquad \qquad ,! 28 (())\text{E: 27} \qquad \qquad \qquad ;
\end{aligned}$$

$$(\lambda(\mathbf{F}^{\wedge}(\mathbf{G}^{\wedge}\mathbf{H}))) = (((\lambda\mathbf{F}) + (\lambda\mathbf{G})) + (\lambda\mathbf{H})), ! 29 (\Rightarrow\text{E: 26,28}) \qquad ;$$

$$\begin{aligned}
& (\theta \mathbf{F} \ \& \ \theta \mathbf{G} \Rightarrow (\lambda(\mathbf{F}^{\wedge}\mathbf{G})) = ((\lambda\mathbf{F})+(\lambda\mathbf{G}))) \\
& \qquad \qquad \qquad ,! 30 (\forall\text{E: P11}) \qquad \qquad \qquad ;
\end{aligned}$$

$$\theta \mathbf{F} \ \& \ \theta \mathbf{G} \Rightarrow (\lambda(\mathbf{F}^{\wedge}\mathbf{G})) = ((\lambda\mathbf{F})+(\lambda\mathbf{G})) \ ,! 31 (())\text{E: 30} \qquad ;$$

$$(\lambda(\mathbf{F}^{\wedge}\mathbf{G})) = ((\lambda\mathbf{F}) + (\lambda\mathbf{G})) \qquad \qquad \qquad ,! 32 (\Rightarrow\text{E: 6,31}) \qquad ;$$

$$(\lambda(\mathbf{F}^{\wedge}(\mathbf{G}^{\wedge}\mathbf{H}))) = ((\lambda(\mathbf{F}^{\wedge}\mathbf{G})) + (\lambda\mathbf{H})) \qquad \qquad \qquad ,! 33 (=E: 29,32) \qquad ;$$

! Line 38 will have multiple citations, hence it is put here. ;

$$\theta (\mathbf{F}^{\wedge}\mathbf{G}) \ \& \ \theta \mathbf{H} \qquad \qquad \qquad ,! 34 (\&\text{I: 5,9}) \qquad \qquad \qquad ;$$

$$\begin{aligned}
& (\theta (\mathbf{F}^{\wedge}\mathbf{G}) \ \& \ \theta \mathbf{H} \Rightarrow (\lambda((\mathbf{F}^{\wedge}\mathbf{G})^{\wedge}\mathbf{H})) = ((\lambda(\mathbf{F}^{\wedge}\mathbf{G}))+(\lambda\mathbf{H}))) \\
& \qquad \qquad \qquad ,! 35 (\forall\text{E: P11}) \qquad \qquad \qquad ;
\end{aligned}$$

$$\begin{aligned}
& \theta (\mathbf{F}^{\wedge}\mathbf{G}) \ \& \ \theta \mathbf{H} \Rightarrow (\lambda((\mathbf{F}^{\wedge}\mathbf{G})^{\wedge}\mathbf{H})) = ((\lambda(\mathbf{F}^{\wedge}\mathbf{G}))+(\lambda\mathbf{H})) \\
& \qquad \qquad \qquad ,! 36 (())\text{E: 35} \qquad \qquad \qquad ;
\end{aligned}$$

$$(\lambda((\mathbf{F}^{\wedge}\mathbf{G})^{\wedge}\mathbf{H})) = ((\lambda(\mathbf{F}^{\wedge}\mathbf{G})) + (\lambda\mathbf{H})) \qquad \qquad \qquad ,! 37 (\Rightarrow\text{E: 34,36}) \qquad ;$$

$$(\lambda((\mathbf{F}^{\wedge}\mathbf{G})^{\wedge}\mathbf{H})) = (\lambda(\mathbf{F}^{\wedge}(\mathbf{G}^{\wedge}\mathbf{H}))) \qquad \qquad \qquad ,! 38 (=E: 33,37) \qquad ;$$

! To prove:

$$\forall i (\leq[1,i] \ \& \ \leq[i,(\lambda(\mathbf{F}^{\wedge}\mathbf{G}))]) \Rightarrow ((\mathbf{F}^{\wedge}(\mathbf{G}^{\wedge}\mathbf{H}))'i) = ((\mathbf{F}^{\wedge}\mathbf{G})'i) \) \qquad ;$$

$$\mathbf{i} \qquad \qquad \qquad ,! 39 (\text{Prem}) \qquad \qquad \qquad ;$$

$$\leq[1,\mathbf{i}] \ \& \ \leq[\mathbf{i},(\lambda(\mathbf{F}^{\wedge}\mathbf{G}))] \qquad \qquad \qquad ,! 40 (\text{Prem}) \qquad \qquad \qquad ;$$

$$\leq[1,\mathbf{i}] \qquad \qquad \qquad ,! 41 (\&\text{E: 40}) \qquad \qquad \qquad ;$$

$$\leq[\mathbf{i},(\lambda(\mathbf{F}^{\wedge}\mathbf{G}))] \qquad \qquad \qquad ,! 42 (\&\text{E: 40}) \qquad \qquad \qquad ;$$

$$\theta \mathbf{H} \ \& \ \leq[1,\mathbf{i}] \ \& \ \leq[\mathbf{i},(\lambda(\mathbf{F}^{\wedge}\mathbf{G}))] \qquad \qquad \qquad ,! 43 (\&\text{I: 5,40}) \qquad \qquad \qquad ;$$

$$\begin{aligned}
& (\theta \mathbf{H} \ \& \ \leq[1,\mathbf{i}] \ \& \ \leq[\mathbf{i},(\lambda(\mathbf{F}^{\wedge}\mathbf{G}))]) \\
& \Rightarrow (((\mathbf{F}^{\wedge}\mathbf{G})^{\wedge}\mathbf{H})'i) = ((\mathbf{F}^{\wedge}\mathbf{G})'i) \)
\end{aligned}$$

$$,! 44 (\forall\text{E: P16}) \qquad \qquad \qquad ;$$

$\theta H \ \& \ \leq[1, i] \ \& \ \leq[i, (\lambda(F^G))]$
 $\Rightarrow (((F^G)^H)^i) = ((F^G)^i)$,! 45 ((E: 44) i
 $((F^G)^H)^i = ((F^G)^i)$,! 46 (\Rightarrow E: 43,45) i
 $(\leq[1, i] \Rightarrow \omega[i])$,! 47 (\forall E: V3.7) i
 $\leq[1, i] \Rightarrow \omega[i]$,! 48 ((E: 47) i
 $\omega[i]$,! 49 (\Rightarrow E: 41,48) i
 $(\theta F \Rightarrow \omega[(\lambda F)])$,! 50 (\forall E: C1.21) i
 $\theta F \Rightarrow \omega[(\lambda F)]$,! 51 ((E: 50) i
 $\omega[(\lambda F)]$,! 52 (\Rightarrow E: 3,51) i
 $\omega[i] \ \& \ \omega[(\lambda F)]$,! 53 (&I: 49,52) i
 $(\omega[i] \ \& \ \omega[(\lambda F)] \Rightarrow \leq[i, (\lambda F)] \vee <[(\lambda F), i])$
, ! 54 (\forall E: V4.43;
 (λF) : C1.17,3) i
 $\omega[i] \ \& \ \omega[(\lambda F)] \Rightarrow \leq[i, (\lambda F)] \vee <[(\lambda F), i]$
, ! 55 ((E: 54) i
 $\leq[i, (\lambda F)] \vee <[(\lambda F), i]$,! 56 (\Rightarrow E: 53,55) i
 $\leq[i, (\lambda F)]$,! 57 (Prem) i
 $\theta (G^H) \ \& \ \leq[1, i]$,! 58 (&I: 13,41) i
 $\theta (G^H) \ \& \ \leq[1, i] \ \& \ \leq[i, (\lambda F)]$,! 59 (&I: 57,58) i
 $(\theta (G^H) \ \& \ \leq[1, i] \ \& \ \leq[i, (\lambda F)]$
 $\Rightarrow ((F^G)^H)^i = (F^i))$
, ! 60 (\forall E: P16) i
 $\theta (G^H) \ \& \ \leq[1, i] \ \& \ \leq[i, (\lambda F)] \Rightarrow ((F^G)^H)^i = (F^i)$
, ! 61 ((E: 60) i
 $((F^G)^H)^i = (F^i)$,! 62 (\Rightarrow E: 59,61) i
 $\theta G \ \& \ \leq[1, i]$,! 63 (&I: 4,41) i
 $\theta G \ \& \ \leq[1, i] \ \& \ \leq[i, (\lambda F)]$,! 64 (&I: 57,63) i
 $(\theta G \ \& \ \leq[1, i] \ \& \ \leq[i, (\lambda F)] \Rightarrow ((F^G)^i) = (F^i))$
, ! 65 (\forall E: P16) i
 $\theta G \ \& \ \leq[1, i] \ \& \ \leq[i, (\lambda F)] \Rightarrow ((F^G)^i) = (F^i)$

,! 66 (())E: 65) i

$((F^G)^i) = (F^i)$,! 67 (\Rightarrow E: 64,66) i

$((F^G)^i) = ((F^G)^i)$,! 68 (=E: 62,67) i

$\leq[i, (\lambda F)] \Rightarrow ((F^G)^i) = ((F^G)^i)$,! 69 (\Rightarrow I: 57,68) i

$\langle [(\lambda F), i]$,! 70 (Prem) i

$\theta G \ \& \ \langle [(\lambda F), i]$,! 71 (&I: 4,70) i

$\theta G \ \& \ \langle [(\lambda F), i] \ \& \ \leq[i, (\lambda(F^G))]$,! 72 (&I: 42,71) i

$(\theta G \ \& \ \langle [(\lambda F), i] \ \& \ \leq[i, (\lambda(F^G))]$
 $\Rightarrow ((F^G)^i) = (G^i - (\lambda F)))$,! 73 (\forall E: P20) i

$\theta G \ \& \ \langle [(\lambda F), i] \ \& \ \leq[i, (\lambda(F^G))]$
 $\Rightarrow ((F^G)^i) = (G^i - (\lambda F))$,! 74 (())E: 73) i

$((F^G)^i) = (G^i - (\lambda F))$,! 75 (\Rightarrow E: 72,74) i

$(\theta(F^G) \ \& \ \theta H \Rightarrow \leq[(\lambda(F^G)), (\lambda((F^G)^H))]$) ,! 76 (\forall E: P12) i

$\theta(F^G) \ \& \ \theta H \Rightarrow \leq[(\lambda(F^G)), (\lambda((F^G)^H))]$,! 77 (())E: 76) i

$\leq[(\lambda(F^G)), (\lambda((F^G)^H))]$,! 78 (\Rightarrow E: 34,77) i

$\leq[(\lambda(F^G)), (\lambda(F^G^H))]$,! 79 (=E: 38,78) i

$\leq[i, (\lambda(F^G))] \ \& \ \leq[(\lambda(F^G)), (\lambda(F^G^H))]$,! 80 (&I: 42,79) i

$(\leq[i, (\lambda(F^G))] \ \& \ \leq[(\lambda(F^G)), (\lambda(F^G^H))]$
 $\Rightarrow \leq[i, (\lambda(F^G^H))])$,! 81 (\forall E: v3.20;
 $(\lambda(F^G))$: C1.17,9;
 $(\lambda(F^G^H))$: C1.17,17) i

$\leq[i, (\lambda(F^G))] \ \& \ \leq[(\lambda(F^G)), (\lambda(F^G^H))]$
 $\Rightarrow \leq[i, (\lambda(F^G^H))]$,! 82 (())E: 81) i

$\leq[i, (\lambda(F^G^H))]$,! 83 (\Rightarrow E: 80,82) i

$\theta(G^H) \ \& \ \langle [(\lambda F), i]$,! 84 (&I: 13,70) i

$\theta(G^H) \ \& \ \langle [(\lambda F), i] \ \& \ \leq[i, (\lambda(F^G^H))]$

,! 85 (&I: 83,84) ;

$$(\theta (G^H) \& \leq [(\lambda F), i] \& \leq [i, (\lambda (F^H))])$$

$$\Rightarrow ((F^H)^i) = ((G^H)^{(i-(\lambda F))})$$

,! 86 ($\forall E$: P20) ;

$$\theta (G^H) \& \leq [(\lambda F), i] \& \leq [i, (\lambda (F^H))]$$

$$\Rightarrow ((F^H)^i) = ((G^H)^{(i-(\lambda F))})$$

,! 87 ($(\)E$: 86) ;

$$((F^H)^i) = ((G^H)^{(i-(\lambda F))})$$

,! 88 ($\Rightarrow E$: 85,87) ;

$$(\leq [(\lambda F), i] \Rightarrow \leq [1, (i-(\lambda F))])$$

,! 89 ($\forall E$: V6.99;
(λF): C1.17,3) ;

$$\leq [(\lambda F), i] \Rightarrow \leq [1, (i-(\lambda F))]$$

,! 90 ($(\)E$: 89) ;

$$\leq [1, (i-(\lambda F))]$$

,! 91 ($\Rightarrow E$: 70,90) ;

$$\theta H \& \leq [1, (i-(\lambda F))]$$

,! 92 (&I: 5,91) ;

$$\leq [i, ((\lambda F)+(\lambda G))]$$

,! 93 ($=E$: 32,42) ;

$$(\leq [(\lambda F), i] \Rightarrow \leq [(\lambda F), i])$$

,! 94 ($\forall E$: V4.5;
(λF): C1.17,3) ;

$$\leq [(\lambda F), i] \Rightarrow \leq [(\lambda F), i]$$

,! 95 ($(\)E$: 94) ;

$$\leq [(\lambda F), i]$$

,! 96 ($\Rightarrow E$: 70,95) ;

$$\leq [i, ((\lambda F)+(\lambda G))] \& \leq [(\lambda F), i]$$

,! 97 (&I: 93,96) ;

$$(\leq [i, ((\lambda F)+(\lambda G))] \& \leq [(\lambda F), i] \Rightarrow \leq [(i-(\lambda F)), (\lambda G)])$$

,! 98 ($\forall E$: V6.83;
(λF): C1.17,3
(λG): C1.17,4) ;

$$\leq [i, ((\lambda F)+(\lambda G))] \& \leq [(\lambda F), i] \Rightarrow \leq [(i-(\lambda F)), (\lambda G)]$$

,! 99 ($(\)E$: 98) ;

$$\leq [(i-(\lambda F)), (\lambda G)]$$

,! 100 ($\Rightarrow E$: 97,99) ;

$$\theta H \& \leq [1, (i-(\lambda F))] \& \leq [(i-(\lambda F)), (\lambda G)]$$

,! 101 (&I: 92,100) ;

$$(\theta H \& \leq [1, (i-(\lambda F))] \& \leq [(i-(\lambda F)), (\lambda G)])$$

$$\Rightarrow ((G^H)^{(i-(\lambda F))}) = (G^i)$$

,! 102 ($\forall E$: P16;
($i-(\lambda F)$): V5.7,96) ;

$$\theta H \& \leq [1, (i-(\lambda F))] \& \leq [(i-(\lambda F)), (\lambda G)]$$

$$\Rightarrow ((G^H)'(i-(\lambda F))) = (G'(i-(\lambda F)))$$

,! 103 (E: 102) ;

$$((G^H)'(i-(\lambda F))) = (G'(i-(\lambda F)))$$

,! 104 (\Rightarrow E: 101,103) ;

$$((F^H)'(i-(\lambda F))) = (G'(i-(\lambda F)))$$

,! 105 (=E: 88,104) ;

$$((F^H)'(i-(\lambda F))) = ((F^G)'i)$$

,! 106 (=E: 75,105) ;

$$\leq[(\lambda F), i] \Rightarrow ((F^H)'(i-(\lambda F))) = ((F^G)'i)$$

,! 107 (\Rightarrow I: 70,106) ;

$$((F^H)'(i-(\lambda F))) = ((F^G)'i)$$

,! 108 (\forall E: 56,69,107) ;

$$\leq[1, i] \ \& \ \leq[i, (\lambda(F^G))] \Rightarrow ((F^H)'(i-(\lambda F))) = ((F^G)'i)$$

,! 109 (\Rightarrow I: 40,108) ;

$$(\leq[1, i] \ \& \ \leq[i, (\lambda(F^G))]) \Rightarrow ((F^H)'(i-(\lambda F))) = ((F^G)'i)$$

,! 110 (I: 109) ;

$$\forall i (\leq[1, i] \ \& \ \leq[i, (\lambda(F^G))]) \Rightarrow ((F^H)'(i-(\lambda F))) = ((F^G)'i)$$

,! 111 (\forall I: 39,110) ;

$$(\lambda(F^H)) = ((\lambda(F^G)) + (\lambda H))$$

$$\& \ \forall i (\leq[1, i] \ \& \ \leq[i, (\lambda(F^G))]) \Rightarrow ((F^H)'(i-(\lambda F))) = ((F^G)'i)$$

,! 112 (&I: 33,111) ;

! To prove:

$$\forall i (\leq[1, i] \ \& \ \leq[i, (\lambda H)])$$

$$\Rightarrow ((F^H)'((\lambda(F^G))+i)) = (H'i)$$

; ;

$$i$$

,! 113 (Prem) ;

$$\leq[1, i] \ \& \ \leq[i, (\lambda H)]$$

,! 114 (Prem) ;

$$\leq[1, i]$$

,! 115 (&E: 114) ;

$$\leq[i, (\lambda H)]$$

,! 116 (&E: 114) ;

$$(\theta F \ \& \ \theta G \Rightarrow \leq[(\lambda F), (\lambda(F^G))])$$

,! 117 (\forall E: P12) ;

$$\theta F \ \& \ \theta G \Rightarrow \leq[(\lambda F), (\lambda(F^G))]$$

,! 118 (E: 117) ;

$$\leq[(\lambda F), (\lambda(F^G))]$$

,! 119 (\Rightarrow E: 6,118) ;

$$\leq[(\lambda F), (\lambda(F^G))] \ \& \ \leq[1, i]$$

,! 120 (&I: 115,119) ;

$$(\leq[(\lambda F), (\lambda(F^G))] \ \& \ \leq[1, i]) \Rightarrow \leq[(\lambda F), ((\lambda(F^G))+i)]$$

,! 121 (\forall E: V4.38) ;

(λF) : C1.17,3;
 $(\lambda(F^G))$: C1.17,9) ;

$\leq[(\lambda F), (\lambda(F^G))] \ \& \ \leq[1, i] \Rightarrow <[(\lambda F), ((\lambda(F^G))+i)]$
, ! 122 ()E: 121) ;

$<[(\lambda F), ((\lambda(F^G))+i)]$
, ! 123 (\Rightarrow E: 120,122) ;

$\theta(G^H) \ \& \ <[(\lambda F), ((\lambda(F^G))+i)]$
, ! 124 ($\&$ I: 13,123) ;

$\omega[(\lambda(F^G))] \ \& \ \omega[i]$
, ! 125 (\top E: v1.7,123) ;

$\omega[(\lambda(F^G))]$
, ! 126 ($\&$ E: 125) ;

$\leq[i, (\lambda H)] \ \& \ \omega[(\lambda(F^G))]$
, ! 127 ($\&$ I: 116,126) ;

$(\leq[i, (\lambda H)] \ \& \ \omega[(\lambda(F^G))])$
 $\Rightarrow \leq[((\lambda(F^G))+i), ((\lambda(F^G))+(\lambda H))]$
, ! 128 (\forall E: v3.29;
 (λH) : C1.17,5;
 $(\lambda(F^G))$: C1.17,9) ;

$\leq[i, (\lambda H)] \ \& \ \omega[(\lambda(F^G))]$
 $\Rightarrow \leq[((\lambda(F^G))+i), ((\lambda(F^G))+(\lambda H))]$
, ! 129 ()E: 128) ;

$\leq[((\lambda(F^G))+i), ((\lambda(F^G))+(\lambda H))]$
, ! 130 (\Rightarrow E: 127,129) ;

$\leq[((\lambda(F^G))+i), (\lambda(F^G^H))]$
, ! 131 (=E: 33,130) ;

$\theta(G^H) \ \& \ <[(\lambda F), ((\lambda(F^G))+i)]$
 $\& \ \leq[((\lambda(F^G))+i), (\lambda(F^G^H))]$
, ! 132 ($\&$ I: 124,131) ;

$(\theta(G^H) \ \& \ <[(\lambda F), ((\lambda(F^G))+i)])$
 $\& \ \leq[((\lambda(F^G))+i), (\lambda(F^G^H))]$
 $\Rightarrow ((F^G^H))'((\lambda(F^G))+i)$
 $= ((G^H))'(((\lambda(F^G))+i)-(\lambda F)))$
, ! 133 (\forall E: P20;
 $((\lambda(F^G))+i)$; v1.7,125) ;

$\theta(G^H) \ \& \ <[(\lambda F), ((\lambda(F^G))+i)]$
 $\& \ \leq[((\lambda(F^G))+i), (\lambda(F^G^H))]$
 $\Rightarrow ((F^G^H))'((\lambda(F^G))+i)$
 $= ((G^H))'(((\lambda(F^G))+i)-(\lambda F)))$
, ! 134 ()E: 133) ;

$$((F \wedge (G \wedge H)) \dot{\ } ((\lambda(F \wedge G)) + i)) = ((G \wedge H) \dot{\ } (((\lambda(F \wedge G)) + i) - (\lambda F)))$$

,! 135 ($\Rightarrow E$: 132,134)
i

$$((F \wedge (G \wedge H)) \dot{\ } ((\lambda(F \wedge G)) + i)) = ((G \wedge H) \dot{\ } (((\lambda(F \wedge G)) + i) - (\lambda F)))$$

,! 135 ($\Rightarrow E$: 132,134)
i

$$\omega[(\lambda F)] \ \& \ \omega[(\lambda G)]$$

,! 136 (TE : v1.7,32)
i

$$\omega[i]$$

,! 137 ($\&E$: 125) ;

$$\omega[(\lambda F)] \ \& \ \omega[(\lambda G)] \ \& \ \omega[i]$$

,! 138 ($\&I$: 136,137)
i

$$(\ \omega[(\lambda F)] \ \& \ \omega[(\lambda G)] \ \& \ \omega[i]$$

$$\Rightarrow (((\lambda F) + (\lambda G)) + i) - (\lambda F) = ((\lambda G) + i)$$

,! 139 ($\forall E$ v6.43;
(λF): C1.17,3;
(λG): C1.17,4) ;

$$\omega[(\lambda F)] \ \& \ \omega[(\lambda G)] \ \& \ \omega[i]$$

$$\Rightarrow (((\lambda F) + (\lambda G)) + i) - (\lambda F) = ((\lambda G) + i)$$

,! 140 ($()E$: 139) ;

$$(((\lambda F) + (\lambda G)) + i) - (\lambda F) = ((\lambda G) + i)$$

,! 141 ($\Rightarrow E$: 138,140)
i

$$(((\lambda(F \wedge G)) + i) - (\lambda F)) = ((\lambda G) + i)$$

,! 142 ($=E$: 32,141) ;

$$((F \wedge (G \wedge H)) \dot{\ } ((\lambda(F \wedge G)) + i)) = ((G \wedge H) \dot{\ } ((\lambda G) + i))$$

,! 143 ($=E$: 135,142)
i

$$\theta \ G \ \& \ \leq[1, i] \ \& \ \leq[i, (\lambda H)]$$

,! 144 ($\&I$: 4,114) ;

$$(\ \theta \ G \ \& \ \leq[1, i] \ \& \ \leq[i, (\lambda H)] \Rightarrow ((G \wedge H) \dot{\ } ((\lambda G) + i)) = (H \dot{\ } i)$$

,! 145 ($\forall E$: P19) ;

$$\theta \ G \ \& \ \leq[1, i] \ \& \ \leq[i, (\lambda H)] \Rightarrow ((G \wedge H) \dot{\ } ((\lambda G) + i)) = (H \dot{\ } i)$$

,! 146 ($()E$: 145) ;

$$((G \wedge H) \dot{\ } ((\lambda G) + i)) = (H \dot{\ } i)$$

,! 147 ($\Rightarrow E$: 144,146)
i

$$((F \wedge (G \wedge H)) \dot{\ } ((\lambda(F \wedge G)) + i)) = (H \dot{\ } i)$$

,! 148 ($=E$: 143,147)
i

$$\leq[1, i] \ \& \ \leq[i, (\lambda H)] \Rightarrow ((F \wedge (G \wedge H)) \dot{\ } ((\lambda(F \wedge G)) + i)) = (H \dot{\ } i)$$

,! 149 ($\Rightarrow I$: 114,148)
i

$$(\leq[1, i] \ \& \ \leq[i, (\lambda H)] \Rightarrow ((F \wedge (G \wedge H))'((\lambda(F \wedge G)) + i)) = (H' i))$$

, ! 150 (()I: 149) ;

$$\forall i (\leq[1, i] \ \& \ \leq[i, (\lambda H)]$$

$$\Rightarrow ((F \wedge (G \wedge H))'((\lambda(F \wedge G)) + i)) = (H' i))$$

, ! 151 (\forall I: 113, 150)

i

$$(\lambda(F \wedge (G \wedge H))) = ((\lambda(F \wedge G)) + (\lambda H))$$

$$\& \ \forall i (\leq[1, i] \ \& \ \leq[i, (\lambda(F \wedge G))] \Rightarrow ((F \wedge (G \wedge H))' i) = ((F \wedge G)' i))$$

$$\& \ \forall i (\leq[1, i] \ \& \ \leq[i, (\lambda H)]$$

$$\Rightarrow ((F \wedge (G \wedge H))'((\lambda(F \wedge G)) + i)) = (H' i))$$

, ! 152 (& I: 112, 151)

i

$$(F \wedge (G \wedge H)) \equiv ((F \wedge G) \wedge H)$$

, ! 153 (\Rightarrow E: 19, 152)

i

$$\theta F \ \& \ \theta G \ \& \ \theta H \Rightarrow (F \wedge (G \wedge H)) \equiv ((F \wedge G) \wedge H)$$

, ! 154 (\Rightarrow I: 2, 153) ;

$$(\theta F \ \& \ \theta G \ \& \ \theta H \Rightarrow (F \wedge (G \wedge H)) \equiv ((F \wedge G) \wedge H))$$

, ! 155 (()I: 154) ;

$$\forall F \forall G \forall H (\theta F \ \& \ \theta G \ \& \ \theta H \Rightarrow (F \wedge (G \wedge H)) \equiv ((F \wedge G) \wedge H))$$

! 156 (\forall I: 1, 155) ;

□

! 33. i

$$\vdash \forall F \forall n \forall m (\leq[m, n] \ \& \ (\lambda F) = n$$

$$\Rightarrow \exists G \exists H ((\lambda G) = m \ \& \ (\lambda H) = (n - m) \ \& \ F \equiv (G \wedge H))) ;$$

$$F, n, m \quad , ! 1 \text{ (Prem)} \quad i$$

$$\leq[m, n] \ \& \ (\lambda F) = n \quad , ! 2 \text{ (Prem)} \quad i$$

$$\leq[m, n] \quad , ! 3 \text{ (\& E)} \quad i$$

$$(\lambda F) = n \quad , ! 4 \text{ (\& E)} \quad i$$

$$\theta F \quad , ! 5 \text{ (T E: C1.17, 4)} \quad i$$

$$(\theta F \Rightarrow f F) \quad , ! 6 \text{ (\forall E: C1.2)} \quad i$$

$$\theta F \Rightarrow f F \quad , ! 7 \text{ (() E: 6)} \quad i$$

$$f F \quad , ! 8 \text{ (\Rightarrow E: 5, 7)} \quad i$$

$$((\lambda F) = n \Rightarrow (F^D) \equiv (1 _ n)) \quad , ! 9 \text{ (\forall E: C1.24)} \quad i$$

$$(\lambda F) = n \Rightarrow (F^D) \equiv (1 _ n) \quad , ! 10 \text{ (() E: 9)} \quad i$$

$$\begin{aligned}
(\mathbf{F}^D) &\equiv (1 _ n) && ,! 11 (\Rightarrow E: 4,10) && ; \\
(\leq[m,n] \ \& \ (\lambda \mathbf{F}) = n \Rightarrow (\lambda(\mathbf{F} \lceil (1 _ m))) = m) &&& ,! 12 (\forall E: C1.38) && ; \\
\leq[m,n] \ \& \ (\lambda \mathbf{F}) = n \Rightarrow (\lambda(\mathbf{F} \lceil (1 _ m))) = m &&& ,! 13 (()E: 12) && ; \\
(\lambda(\mathbf{F} \lceil (1 _ m))) = m &&& ,! 14 (\Rightarrow E: 2,13) && ; \\
(\leq[m,n] \ \& \ (\lambda \mathbf{F}) = n \\
\Rightarrow (\lambda((\Theta_m) \circ (\mathbf{F} \lceil ((m+1) _ n)))) = (n-m)) &&& ,! 15 (\forall E: C1.54) && ; \\
\leq[m,n] \ \& \ (\lambda \mathbf{F}) = n \\
\Rightarrow (\lambda((\Theta_m) \circ (\mathbf{F} \lceil ((m+1) _ n)))) = (n-m) &&& ,! 16 (()E: 15) && ; \\
(\lambda((\Theta_m) \circ (\mathbf{F} \lceil ((m+1) _ n)))) = (n-m) &&& ,! 17 (\Rightarrow E: 2,16) && ; \\
(\lambda(\mathbf{F} \lceil (1 _ m))) = m \\
\& \ (\lambda((\Theta_m) \circ (\mathbf{F} \lceil ((m+1) _ n)))) = (n-m) &&& ,! 18 (\&I: 14,17) && ; \\
(\leq[m,n] \Rightarrow \omega[m]) &&& ,! 19 (\forall E: V3.6) && ; \\
\leq[m,n] \Rightarrow \omega[m] &&& ,! 20 (()E: 19) && ; \\
\omega[m] &&& ,! 21 (\Rightarrow E: 3,20) && ; \\
(\omega[m] \Rightarrow ((\Theta_m)^*) \equiv (\Theta_m)) &&& ,! 22 (\forall E: V10.24) && ; \\
\omega[m] \Rightarrow ((\Theta_m)^*) \equiv (\Theta_m) &&& ,! 23 (()E: 22) && ; \\
((\Theta_m)^*) \equiv (\Theta_m) &&& ,! 24 (\Rightarrow E: 21,23) && ; \\
((\Theta_m)^*) \equiv (\Theta_m) \Rightarrow (\Theta_m) \equiv ((\Theta_m)^*) &&& ,! 25 (\forall E: III1.8) && ; \\
((\Theta_m)^*) \equiv (\Theta_m) \Rightarrow (\Theta_m) \equiv ((\Theta_m)^*) &&& ,! 26 (()E: 25) && ; \\
(\Theta_m) \equiv ((\Theta_m)^*) &&& ,! 27 (\Rightarrow E: 24,26) && ; \\
((\mathbf{F} \lceil ((m+1) _ n))^D) \subseteq ((m+1) _ n) &&& ,! 28 (\forall E: III7.29) && ; \\
(\omega[m] \Rightarrow ((m+1) _ n) \subseteq (m \infty)) &&& ,! 29 (\forall E: VI2.42) && ; \\
\omega[m] \Rightarrow ((m+1) _ n) \subseteq (m \infty) &&& ,! 30 (()E: 29) && ; \\
((m+1) _ n) \subseteq (m \infty) &&& ,! 31 (\Rightarrow E: 21,30) && ;
\end{aligned}$$

$$\begin{aligned} &\Rightarrow ((\ominus m) \circ ((\oplus m) \circ (F \uparrow ((m+1) _ n)))) \\ &\quad \equiv (F \uparrow ((m+1) _ n)) \end{aligned} \quad ,! 49 (())E: 48) \quad ;$$

$$\begin{aligned} ((\ominus m) \circ ((\oplus m) \circ (F \uparrow ((m+1) _ n)))) &\equiv (F \uparrow ((m+1) _ n)) \\ &\quad ,! 50 (\Rightarrow E: 47,49) \quad ; \end{aligned}$$

$$\begin{aligned} ((\ominus m) \circ ((\oplus m) \circ (F \uparrow ((m+1) _ n)))) &\equiv (F \uparrow ((m+1) _ n)) \\ \Rightarrow ((F \uparrow (1 _ m)) &\sqcup ((\ominus m) \circ ((\oplus m) \circ (F \uparrow ((m+1) _ n))))) \\ &\equiv ((F \uparrow (1 _ m)) \sqcup (F \uparrow ((m+1) _ n))) \end{aligned} \quad ,! 51 (\forall E: III2.13) \quad ;$$

$$\begin{aligned} ((\ominus m) \circ ((\oplus m) \circ (F \uparrow ((m+1) _ n)))) &\equiv (F \uparrow ((m+1) _ n)) \\ \Rightarrow ((F \uparrow (1 _ m)) &\sqcup ((\ominus m) \circ ((\oplus m) \circ (F \uparrow ((m+1) _ n))))) \\ &\equiv ((F \uparrow (1 _ m)) \sqcup (F \uparrow ((m+1) _ n))) \end{aligned} \quad ,! 52 (())E: 51) \quad ;$$

$$\begin{aligned} ((F \uparrow (1 _ m)) &\sqcup ((\ominus m) \circ ((\oplus m) \circ (F \uparrow ((m+1) _ n))))) \\ &\equiv ((F \uparrow (1 _ m)) \sqcup (F \uparrow ((m+1) _ n))) \end{aligned} \quad ,! 53 (\Rightarrow E: 50,52) \quad ;$$

$$\begin{aligned} ((F \uparrow (1 _ m)) &\sqcup ((\ominus (\lambda(F \uparrow (1 _ m))) \circ ((\oplus m) \circ (F \uparrow ((m+1) _ n))))) \\ &\equiv ((F \uparrow (1 _ m)) \sqcup (F \uparrow ((m+1) _ n))) \end{aligned} \quad ,! 54 (=E: 14,53) \quad ;$$

$$\begin{aligned} ((F \uparrow (1 _ m))^{\wedge}((\oplus m) \circ (F \uparrow ((m+1) _ n)))) & \\ &\equiv ((F \uparrow (1 _ m)) \sqcup (F \uparrow ((m+1) _ n))) \end{aligned} \quad ,! 55 (DI: P1,54) \quad ;$$

$$\omega[1] \& \omega[m] \quad ,! 56 (&I: IV9.2,21) \quad ;$$

$$(\omega[1] \& \omega[m] \Rightarrow \leq[1, (m+1)]) \quad ,! 57 (\forall E: V3.34) \quad ;$$

$$\omega[1] \& \omega[m] \Rightarrow \leq[1, (m+1)] \quad ,! 58 (())E: 57) \quad ;$$

$$\leq[1, (m+1)] \quad ,! 59 (\Rightarrow E: 56,58) \quad ;$$

$$\leq[1, (m+1)] \& \leq[m, n] \quad ,! 60 (&I: 3,59) \quad ;$$

$$\begin{aligned} (\leq[1, (m+1)] \& \leq[m, n]) \\ \Rightarrow ((1 _ m) \cup ((m+1) _ n)) &\equiv (1 _ n) \end{aligned} \quad ,! 61 (\forall E: VI2.29) \quad ;$$

$$\begin{aligned} \leq[1, (m+1)] \& \leq[m, n] \\ \Rightarrow ((1 _ m) \cup ((m+1) _ n)) &\equiv (1 _ n) \end{aligned}$$

,! 62 (E: 61) ;

$((1 _ m) \cup ((m+1) _ n)) \equiv (1 _ n)$,! 63 (\Rightarrow E: 60,62) ;

$(F^D) \equiv (1 _ n) \ \& \ ((1 _ m) \cup ((m+1) _ n)) \equiv (1 _ n)$
,! 64 (&I: 11,63) ;

$((F^D) \equiv (1 _ n) \ \& \ ((1 _ m) \cup ((m+1) _ n)) \equiv (1 _ n))$
 $\Rightarrow (F^D) \equiv ((1 _ m) \cup ((m+1) _ n))$
,! 65 (\forall E: III1.17) ;

$(F^D) \equiv (1 _ n) \ \& \ ((1 _ m) \cup ((m+1) _ n)) \equiv (1 _ n)$
 $\Rightarrow (F^D) \equiv ((1 _ m) \cup ((m+1) _ n))$
,! 66 (E: 65) ;

$(F^D) \equiv ((1 _ m) \cup ((m+1) _ n))$,! 67 (\Rightarrow E: 64,66) ;

$((F^D) \equiv ((1 _ m) \cup ((m+1) _ n)))$
 $\Rightarrow ((F \lceil (1 _ m)) \sqcup (F \lceil ((m+1) _ n))) \equiv F$
,! 68 (\forall E: III7.27) ;

$(F^D) \equiv ((1 _ m) \cup ((m+1) _ n))$
 $\Rightarrow ((F \lceil (1 _ m)) \sqcup (F \lceil ((m+1) _ n))) \equiv F$
,! 69 (E: 68) ;

$((F \lceil (1 _ m)) \sqcup (F \lceil ((m+1) _ n))) \equiv F$
,! 70 (\Rightarrow E: 67,69) ;

$((F \lceil (1 _ m))^{\wedge}((\oplus m) \circ (F \lceil ((m+1) _ n))))$
 $\equiv ((F \lceil (1 _ m)) \sqcup (F \lceil ((m+1) _ n)))$
 $\& ((F \lceil (1 _ m)) \sqcup (F \lceil ((m+1) _ n))) \equiv F$
,! 71 (&I: 55,70) ;

$(((F \lceil (1 _ m))^{\wedge}((\oplus m) \circ (F \lceil ((m+1) _ n)))))$
 $\equiv ((F \lceil (1 _ m)) \sqcup (F \lceil ((m+1) _ n)))$
 $\& ((F \lceil (1 _ m)) \sqcup (F \lceil ((m+1) _ n))) \equiv F$
 $\Rightarrow F \equiv ((F \lceil (1 _ m))^{\wedge}((\oplus m) \circ (F \lceil ((m+1) _ n)))))$
,! 72 (\forall E: III1.15) ;

$((F \lceil (1 _ m))^{\wedge}((\oplus m) \circ (F \lceil ((m+1) _ n))))$
 $\equiv ((F \lceil (1 _ m)) \sqcup (F \lceil ((m+1) _ n)))$
 $\& ((F \lceil (1 _ m)) \sqcup (F \lceil ((m+1) _ n))) \equiv F$
 $\Rightarrow F \equiv ((F \lceil (1 _ m))^{\wedge}((\oplus m) \circ (F \lceil ((m+1) _ n))))$
,! 73 (E: 72) ;

$F \equiv ((F \lceil (1 _ m))^{\wedge}((\oplus m) \circ (F \lceil ((m+1) _ n))))$
,! 74 (\Rightarrow E: 71,73) ;

$(\lambda(F \lceil (1 _ m))) = m$

$$\begin{aligned} & \& (\lambda ((\oplus m) \circ (F \uparrow ((m+1) _ n)))) = (n-m) \\ & \& F \equiv ((F \uparrow (1 _ m))^{\wedge} ((\oplus m) \circ (F \uparrow ((m+1) _ n)))) \\ & \hspace{15em}, ! 75 (\&I: 18,74) \quad i \end{aligned}$$

$$\begin{aligned} & ((\lambda (F \uparrow (1 _ m))) = m \\ & \& (\lambda ((\oplus m) \circ (F \uparrow ((m+1) _ n)))) = (n-m) \\ & \& F \equiv ((F \uparrow (1 _ m))^{\wedge} ((\oplus m) \circ (F \uparrow ((m+1) _ n)))) \\ & \hspace{15em}, ! 76 ((I: 75) \quad i \end{aligned}$$

$$\begin{aligned} \exists H (& (\lambda (F \uparrow (1 _ m))) = m \\ & \& (\lambda H) = (n-m) \\ & \& F \equiv ((F \uparrow (1 _ m))^{\wedge} H)) \\ & \hspace{15em}, ! 77 (\exists I: 76) \quad i \end{aligned}$$

$$\begin{aligned} \exists G \exists H (& (\lambda G) = m \\ & \& (\lambda H) = (n-m) \\ & \& F \equiv (G^{\wedge} H)) \\ & \hspace{15em}, ! 78 (\exists I: 77) \quad i \end{aligned}$$

$$\begin{aligned} \leq [m, n] \& (\lambda F) = n \\ \Rightarrow \exists G \exists H (& (\lambda G) = m \& (\lambda H) = (n-m) \& F \equiv (G^{\wedge} H)) \\ & \hspace{15em}, ! 79 (\Rightarrow I: 2,78) \quad i \end{aligned}$$

$$\begin{aligned} (\leq [m, n] \& (\lambda F) = n \\ \Rightarrow \exists G \exists H (& (\lambda G) = m \& (\lambda H) = (n-m) \& F \equiv (G^{\wedge} H))) \\ & \hspace{15em}, ! 80 ((I: 79) \quad i \end{aligned}$$

$$\begin{aligned} \forall F \forall n \forall m (\leq [m, n] \& (\lambda F) = n \\ \Rightarrow \exists G \exists H (& (\lambda G) = m \& (\lambda H) = (n-m) \& F \equiv (G^{\wedge} H))) \\ & \hspace{15em}! 81 (\forall I: 1,80) \quad i \end{aligned}$$

□

! 34. i

$$\begin{aligned} \vdash \forall F \forall n \forall m (\leq [m, n] \& (\lambda F) = n \\ \Rightarrow \exists G \exists H (& (\lambda G) = (n-m) \& (\lambda H) = m \& F \equiv (G^{\wedge} H))) \quad i \end{aligned}$$

$$F, n, m \hspace{10em}, ! 1 (\text{Prem}) \quad i$$

$$\leq [m, n] \& (\lambda F) = n \hspace{10em}, ! 2 (\text{Prem}) \quad i$$

$$\leq [m, n] \hspace{10em}, ! 3 (\&E: 2) \quad i$$

$$(\lambda F) = n \hspace{10em}, ! 4 (\&E: 2) \quad i$$

$$(\leq [m, n] \Rightarrow \leq [(n-m), n]) \hspace{10em}, ! 5 (\forall E: \forall 6.74) \quad i$$

$$\leq [m, n] \Rightarrow \leq [(n-m), n] \hspace{10em}, ! 6 ((E: 5) \quad i$$

$$\leq [(n-m), n] \hspace{10em}, ! 7 (\Rightarrow E: 3,6) \quad i$$

$\leq [(\mathbf{n}-\mathbf{m}), \mathbf{n}] \ \& \ (\lambda \mathbf{F}) = \mathbf{n}$,! 8 (&I: 4,7) ;

($\leq [(\mathbf{n}-\mathbf{m}), \mathbf{n}] \ \& \ (\lambda \mathbf{F}) = \mathbf{n}$
 $\Rightarrow \exists \mathbf{G} \exists \mathbf{H} ((\lambda \mathbf{G}) = (\mathbf{n}-\mathbf{m}) \ \& \ (\lambda \mathbf{H}) = (\mathbf{n}-(\mathbf{n}-\mathbf{m})) \ \& \ \mathbf{F} \equiv (\mathbf{G} \wedge \mathbf{H}))$))
,! 9 ($\forall \mathbf{E}$: P33;
 $(\mathbf{n}-\mathbf{m})$: V5.7,3) ;

$\leq [(\mathbf{n}-\mathbf{m}), \mathbf{n}] \ \& \ (\lambda \mathbf{F}) = \mathbf{n}$
 $\Rightarrow \exists \mathbf{G} \exists \mathbf{H} ((\lambda \mathbf{G}) = (\mathbf{n}-\mathbf{m}) \ \& \ (\lambda \mathbf{H}) = (\mathbf{n}-(\mathbf{n}-\mathbf{m})) \ \& \ \mathbf{F} \equiv (\mathbf{G} \wedge \mathbf{H}))$
,! 10 ($()\mathbf{E}$: 9) ;

$\exists \mathbf{G} \exists \mathbf{H} ((\lambda \mathbf{G}) = (\mathbf{n}-\mathbf{m}) \ \& \ (\lambda \mathbf{H}) = (\mathbf{n}-(\mathbf{n}-\mathbf{m})) \ \& \ \mathbf{F} \equiv (\mathbf{G} \wedge \mathbf{H}))$
,! 11 ($\Rightarrow \mathbf{E}$: 8,10) ;

($\leq [\mathbf{m}, \mathbf{n}] \Rightarrow (\mathbf{n}-(\mathbf{n}-\mathbf{m})) = \mathbf{m}$) ,! 12 ($\forall \mathbf{E}$: V6.37) ;

$\leq [\mathbf{m}, \mathbf{n}] \Rightarrow (\mathbf{n}-(\mathbf{n}-\mathbf{m})) = \mathbf{m}$,! 13 ($()\mathbf{E}$: 12) ;

$(\mathbf{n}-(\mathbf{n}-\mathbf{m})) = \mathbf{m}$,! 14 ($\Rightarrow \mathbf{E}$: 3,13) ;

$\exists \mathbf{G} \exists \mathbf{H} ((\lambda \mathbf{G}) = (\mathbf{n}-\mathbf{m}) \ \& \ (\lambda \mathbf{H}) = \mathbf{m} \ \& \ \mathbf{F} \equiv (\mathbf{G} \wedge \mathbf{H}))$
,! 15 ($=\mathbf{E}$: 11,14) ;

$\leq [\mathbf{m}, \mathbf{n}] \ \& \ (\lambda \mathbf{F}) = \mathbf{n}$
 $\Rightarrow \exists \mathbf{G} \exists \mathbf{H} ((\lambda \mathbf{G}) = (\mathbf{n}-\mathbf{m}) \ \& \ (\lambda \mathbf{H}) = \mathbf{m} \ \& \ \mathbf{F} \equiv (\mathbf{G} \wedge \mathbf{H}))$
,! 16 ($\Rightarrow \mathbf{I}$: 2,15) ;

($\leq [\mathbf{m}, \mathbf{n}] \ \& \ (\lambda \mathbf{F}) = \mathbf{n}$
 $\Rightarrow \exists \mathbf{G} \exists \mathbf{H} ((\lambda \mathbf{G}) = (\mathbf{n}-\mathbf{m}) \ \& \ (\lambda \mathbf{H}) = \mathbf{m} \ \& \ \mathbf{F} \equiv (\mathbf{G} \wedge \mathbf{H}))$)
,! 17 ($()\mathbf{I}$: 16) ;

$\forall \mathbf{F} \forall \mathbf{n} \forall \mathbf{m} (\leq [\mathbf{m}, \mathbf{n}] \ \& \ (\lambda \mathbf{F}) = \mathbf{n}$
 $\Rightarrow \exists \mathbf{G} \exists \mathbf{H} ((\lambda \mathbf{G}) = (\mathbf{n}-\mathbf{m}) \ \& \ (\lambda \mathbf{H}) = \mathbf{m} \ \& \ \mathbf{F} \equiv (\mathbf{G} \wedge \mathbf{H}))$)
! 18 ($\forall \mathbf{I}$: 1,17) ;

□

! 35. ;

$\vdash \forall \mathbf{F} \forall \mathbf{n} (\leq [1, \mathbf{n}] \ \& \ (\lambda \mathbf{F}) = \mathbf{n}$
 $\Rightarrow \exists \mathbf{g} \exists \mathbf{H} ((\lambda \mathbf{H}) = (\mathbf{n}-1) \ \& \ \mathbf{F} \equiv ((1 \ \square \ \mathbf{g}) \wedge \mathbf{H}))$)

\mathbf{F}, \mathbf{n} ,! 1 (Prem) ;

$\leq [1, \mathbf{n}] \ \& \ (\lambda \mathbf{F}) = \mathbf{n}$,! 2 (Prem) ;

($\leq [1, \mathbf{n}] \ \& \ (\lambda \mathbf{F}) = \mathbf{n}$
 $\Rightarrow \exists \mathbf{G} \exists \mathbf{H} ((\lambda \mathbf{G}) = 1 \ \& \ (\lambda \mathbf{H}) = (\mathbf{n}-1) \ \& \ \mathbf{F} \equiv (\mathbf{G} \wedge \mathbf{H}))$)
,! 3 ($\forall \mathbf{E}$: P33) ;

$\leq [1, \mathbf{n}] \ \& \ (\lambda \mathbf{F}) = \mathbf{n}$
 $\Rightarrow \exists \mathbf{G} \exists \mathbf{H} ((\lambda \mathbf{G}) = 1 \ \& \ (\lambda \mathbf{H}) = (\mathbf{n}-1) \ \& \ \mathbf{F} \equiv (\mathbf{G} \wedge \mathbf{H}))$
,! 4 ($()\mathbf{E}$: 3) ;

$\exists g \exists H ((\lambda G) = 1 \ \& \ (\lambda H) = (n-1) \ \& \ F \equiv (G^H))$
, ! 5 ($\Rightarrow E: 2,4$) ;

$\exists H ((\lambda G) = 1 \ \& \ (\lambda H) = (n-1) \ \& \ F \equiv (G^H))$
, ! 6 ($\exists E: 5$) ;

$((\lambda G) = 1 \ \& \ (\lambda H) = (n-1) \ \& \ F \equiv (G^H))$
, ! 7 ($\exists E: 6$) ;

$(\lambda G) = 1 \ \& \ (\lambda H) = (n-1) \ \& \ F \equiv (G^H)$, ! 8 ($() E: 7$) ;

$(\lambda G) = 1$, ! 9 ($\& E: 8$) ;

$(\lambda H) = (n-1)$, ! 10 ($\& E: 8$) ;

$F \equiv (G^H)$, ! 11 ($\& E: 8$) ;

θG , ! 12 ($\top E: C1.17,9$) ;

$\theta G \ \& \ F \equiv (G^H)$, ! 13 ($\& I: 11,12$) ;

$((\lambda G) = 1 \Rightarrow \exists b \ G \equiv (1 \ \blacksquare \ b))$, ! 14 ($\forall E: C1.52$) ;

$(\lambda G) = 1 \Rightarrow \exists b \ G \equiv (1 \ \blacksquare \ b)$, ! 15 ($() E: 14$) ;

$\exists b \ G \equiv (1 \ \blacksquare \ b)$, ! 16 ($\Rightarrow E: 9,15$) ;

$G \equiv (1 \ \blacksquare \ g)$, ! 17 ($\exists E: 16$) ;

$\theta G \ \& \ G \equiv (1 \ \blacksquare \ g) \ \& \ F \equiv (G^H)$, ! 18 ($\& I: 13,17$) ;

$(\theta G \ \& \ G \equiv (1 \ \blacksquare \ g) \ \& \ F \equiv (G^H) \Rightarrow F \equiv ((1 \ \blacksquare \ g)^H))$
, ! 19 ($\forall E: P2$) ;

$\theta G \ \& \ G \equiv (1 \ \blacksquare \ g) \ \& \ F \equiv (G^H) \Rightarrow F \equiv ((1 \ \blacksquare \ g)^H)$
, ! 20 ($() E: 19$) ;

$F \equiv ((1 \ \blacksquare \ g)^H)$, ! 21 ($\Rightarrow E: 18,20$) ;

$(\lambda H) = (n-1) \ \& \ F \equiv ((1 \ \blacksquare \ g)^H)$, ! 22 ($\& I: 10,21$) ;

$((\lambda H) = (n-1) \ \& \ F \equiv ((1 \ \blacksquare \ g)^H))$, ! 23 ($() I: 22$) ;

$\exists H ((\lambda H) = (n-1) \ \& \ F \equiv ((1 \ \blacksquare \ g)^H))$, ! 24 ($\exists I: 23$) ;

$\exists g \exists H ((\lambda H) = (n-1) \ \& \ F \equiv ((1 \ \blacksquare \ g)^H))$
, ! 25 ($\exists I: 24$) ;

$\leq [1, n] \ \& \ (\lambda F) = n \Rightarrow \exists g \exists H ((\lambda H) = (n-1) \ \& \ F \equiv ((1 \ \blacksquare \ g)^H))$
, ! 26 ($\Rightarrow I: 2,25$) ;

$(\leq [1, n] \ \& \ (\lambda F) = n$

$\Rightarrow \exists g \exists H ((\lambda H) = (n-1) \ \& \ F \equiv ((1 \ \blacksquare \ g)^H))$
, ! 27 (()I: 26) i

$\forall F \forall n (\leq[1, n] \ \& \ (\lambda F) = n$
 $\Rightarrow \exists g \exists H ((\lambda H) = (n-1) \ \& \ F \equiv ((1 \ \blacksquare \ g)^H))$
! 28 (\forall I: 1, 27) i

□

! 36. i

$\vdash \forall F \forall a \forall n ((F'1) = a \ \& \ (\lambda F) = n$
 $\Rightarrow \exists H ((\lambda H) = (n-1) \ \& \ F \equiv ((1 \ \blacksquare \ a)^H))$ i

F, a, n , ! 1 (Prem) i

$(F'1) = a \ \& \ (\lambda F) = n$, ! 2 (Prem) i

$(F'1) = a$, ! 3 (&E: 2) i

$(\lambda F) = n$, ! 4 (&E: 2) i

$\theta \ F$, ! 5 (TE: C1.17, 4) i

$\theta \ F \ \& \ (F'1) = a$, ! 6 (&I: 3, 5) i

$(\theta \ F \ \& \ (F'1) = a \Rightarrow \leq[1, (\lambda F)])$, ! 7 (VE: C1.30) i

$\theta \ F \ \& \ (F'1) = a \Rightarrow \leq[1, (\lambda F)]$, ! 8 (()E: 7) i

$\leq[1, (\lambda F)]$, ! 9 (\Rightarrow E: 6, 8) i

$\leq[1, n]$, ! 10 (=E: 4, 9) i

$\leq[1, n] \ \& \ (\lambda F) = n$, ! 11 (&I: 4, 10) i

$(\leq[1, n] \ \& \ (\lambda F) = n$

$\Rightarrow \exists g \exists H ((\lambda H) = (n-1) \ \& \ F \equiv ((1 \ \blacksquare \ g)^H))$
, ! 12 (\forall E: P35) i

$\leq[1, n] \ \& \ (\lambda F) = n$

$\Rightarrow \exists g \exists H ((\lambda H) = (n-1) \ \& \ F \equiv ((1 \ \blacksquare \ g)^H))$
, ! 13 (()E: 12) i

$\exists g \exists H ((\lambda H) = (n-1) \ \& \ F \equiv ((1 \ \blacksquare \ g)^H))$
, ! 14 (\Rightarrow E: 11, 13) i

$\exists H ((\lambda H) = (n-1) \ \& \ F \equiv ((1 \ \blacksquare \ g)^H))$, ! 15 (\exists E: 14) i

$((\lambda H) = (n-1) \ \& \ F \equiv ((1 \ \blacksquare \ g)^H))$, ! 16 (\exists E: 15) i

$(\lambda H) = (n-1) \ \& \ F \equiv ((1 \ \blacksquare \ g)^H)$, ! 17 (()E: 16) i

$(\lambda H) = (n-1)$, ! 18 (&E: 17) i

$F \equiv ((1 \cdot g)^H)$,! 19 (&E: 17) ;

$F \equiv ((1 \cdot g)^H) \ \& \ (F'1) = a$,! 20 (&I: 3,19) ;

θH ,! 21 (TE: C1.17,18) ;

$(\theta H \Rightarrow (((1 \cdot g)^H)'1) = g)$,! 22 (\forall E: P23) ;

$\theta H \Rightarrow (((1 \cdot g)^H)'1) = g$,! 23 ($()$ E: 22) ;

$(((1 \cdot g)^H)'1) = g$,! 24 (\Rightarrow E: 21,23) ;

$F \equiv ((1 \cdot g)^H) \ \& \ (F'1) = a \ \& \ (((1 \cdot g)^H)'1) = g$,! 25 (&I: 19,24) ;

$(F \equiv ((1 \cdot g)^H) \ \& \ (F'1) = a \ \& \ (((1 \cdot g)^H)'1) = g$
 $\Rightarrow a = g)$,! 26 (\forall E: III8.29) ;

$F \equiv ((1 \cdot g)^H) \ \& \ (F'1) = a \ \& \ (((1 \cdot g)^H)'1) = g$
 $\Rightarrow a = g$,! 27 ($()$ E: 26) ;

$a = g$,! 28 (\Rightarrow E: 25,27) ;

$\exists H (\lambda H) = (n-1) \ \& \ F \equiv ((1 \cdot a)^H)$,! 29 (=E: 15,28) ;

$(F'1) = a \ \& \ (\lambda F) = n$
 $\Rightarrow \exists H (\lambda H) = (n-1) \ \& \ F \equiv ((1 \cdot a)^H)$,! 30 (\Rightarrow I: 2,29) ;

$(F'1) = a \ \& \ (\lambda F) = n$
 $\Rightarrow \exists H (\lambda H) = (n-1) \ \& \ F \equiv ((1 \cdot a)^H)$,! 31 ($()$ I: 30) ;

$\forall F \forall a \forall n ((F'1) = a \ \& \ (\lambda F) = n$
 $\Rightarrow \exists H (\lambda H) = (n-1) \ \& \ F \equiv ((1 \cdot a)^H)$) ! 32 (\forall I: 1,31) ;

□

! 37. ;

$\vdash \forall F \forall n (\leq[1,n] \ \& \ (\lambda F) = n$
 $\Rightarrow \exists G \exists h (\lambda G) = (n-1) \ \& \ F \equiv (G \wedge (1 \cdot h))$) ;

F, n ,! 1 (Prem) ;

$\leq[1,n] \ \& \ (\lambda F) = n$,! 2 (Prem) ;

$(\leq[1,n] \ \& \ (\lambda F) = n$
 $\Rightarrow \exists G \exists H (\lambda G) = (n-1) \ \& \ (\lambda H) = 1 \ \& \ F \equiv (G \wedge H)$) ;

, ! 3 ($\forall E$: P34) i

$\leq [1, n] \ \& \ (\lambda F) = n$
 $\Rightarrow \exists G \exists H \ (\ (\lambda G) = (n-1) \ \& \ (\lambda H) = 1 \ \& \ F \equiv (G \wedge H) \)$
, ! 4 ($()E$: 3) i

$\exists G \exists H \ (\ (\lambda G) = (n-1) \ \& \ (\lambda H) = 1 \ \& \ F \equiv (G \wedge H) \)$
, ! 5 ($\Rightarrow E$: 2,4) i

$\exists H \ (\ (\lambda G) = (n-1) \ \& \ (\lambda H) = 1 \ \& \ F \equiv (G \wedge H) \)$
, ! 6 ($\exists E$: 5) i

$(\ (\lambda G) = (n-1) \ \& \ (\lambda H) = 1 \ \& \ F \equiv (G \wedge H) \)$
, ! 7 ($\exists E$: 6) i

$(\lambda G) = (n-1) \ \& \ (\lambda H) = 1 \ \& \ F \equiv (G \wedge H)$, ! 8 ($()E$: 7) i

$(\lambda G) = (n-1)$, ! 9 ($\&E$: 8) i

$(\lambda H) = 1$, ! 10 ($\&E$: 8) i

$F \equiv (G \wedge H)$, ! 11 ($\&E$: 8) i

$(\ (\lambda H) = 1 \ \Rightarrow \exists b \ H \equiv (1 \ \square \ b) \)$, ! 12 ($\forall E$: C1.52) i

$(\lambda H) = 1 \ \Rightarrow \exists b \ H \equiv (1 \ \square \ b)$, ! 13 ($()E$: 12) i

$\exists b \ H \equiv (1 \ \square \ b)$, ! 14 ($\Rightarrow E$: 10,13) i

$H \equiv (1 \ \square \ h)$, ! 15 ($\exists E$: 14) i

$H \equiv (1 \ \square \ h) \ \& \ F \equiv (G \wedge H)$, ! 16 ($\&I$: 11,15) i

$(\ H \equiv (1 \ \square \ h) \ \& \ F \equiv (G \wedge H) \ \Rightarrow \ F \equiv (G \wedge (1 \ \square \ h)) \)$
, ! 17 ($\forall E$: P7) i

$H \equiv (1 \ \square \ h) \ \& \ F \equiv (G \wedge H) \ \Rightarrow \ F \equiv (G \wedge (1 \ \square \ h))$
, ! 18 ($()E$: 17) i

$F \equiv (G \wedge (1 \ \square \ h))$, ! 19 ($\Rightarrow E$: 16,18) i

$(\lambda G) = (n-1) \ \& \ F \equiv (G \wedge (1 \ \square \ h))$, ! 20 ($\&I$: 9,19) i

$(\ (\lambda G) = (n-1) \ \& \ F \equiv (G \wedge (1 \ \square \ h)) \)$, ! 21 ($()I$: 20) i

$\exists h \ (\ (\lambda G) = (n-1) \ \& \ F \equiv (G \wedge (1 \ \square \ h)) \)$, ! 22 ($\exists I$: 21) i

$\exists G \exists h \ (\ (\lambda G) = (n-1) \ \& \ F \equiv (G \wedge (1 \ \square \ h)) \)$
, ! 23 ($\exists I$: 22) i

$\leq [1, n] \ \& \ (\lambda F) = n \ \Rightarrow \exists G \exists h \ (\ (\lambda G) = (n-1) \ \& \ F \equiv (G \wedge (1 \ \square \ h)) \)$
, ! 24 ($\Rightarrow I$: 2,23) i

$(\leq[1, n] \ \& \ (\lambda F) = n$
 $\Rightarrow \exists G \exists h ((\lambda G) = (n-1) \ \& \ F \equiv (G^{(1 \ \square \ h)})))$
, ! 25 (()I: 24) i

$\forall F \forall n (\leq[1, n] \ \& \ (\lambda F) = n$
 $\Rightarrow \exists G \exists h ((\lambda G) = (n-1) \ \& \ F \equiv (G^{(1 \ \square \ h)})))$
! 26 (\forall I: 1, 25) i

□

! 38. i

$\vdash \forall a \forall b \exists F ((\lambda F) = 2 \ \& \ (F'1) = a \ \& \ (F'2) = b)$ i

a, b , ! 1 (Prem) i

$\exists F ((\lambda F) = 1 \ \& \ (F'1) = a)$, ! 2 (\forall E: C1.53) i

$((\lambda F) = 1 \ \& \ (F'1) = a)$, ! 3 (\exists E: 2) i

$(\lambda F) = 1 \ \& \ (F'1) = a$, ! 4 (()E: 3) i

$(\lambda F) = 1$, ! 5 (&E: 4) i

$(F'1) = a$, ! 6 (&E: 4) i

$((\lambda F) = 1$
 $\Rightarrow (\lambda(F^{(1 \ \square \ b)})) = (1+1) \ \& \ ((F^{(1 \ \square \ b)})^{(1+1)}) = b)$
, ! 7 (\forall E: P29) i

$(\lambda F) = 1$
 $\Rightarrow (\lambda(F^{(1 \ \square \ b)})) = (1+1) \ \& \ ((F^{(1 \ \square \ b)})^{(1+1)}) = b$
, ! 8 (()E: 7) i

$(\lambda(F^{(1 \ \square \ b)})) = (1+1) \ \& \ ((F^{(1 \ \square \ b)})^{(1+1)}) = b$
, ! 9 (\Rightarrow E: 5, 8) i

$(\lambda(F^{(1 \ \square \ b)})) = 2 \ \& \ ((F^{(1 \ \square \ b)})^{(2)}) = b$
, ! 10 (=E: V2.69, 9) i

$\theta (1 \ \square \ b)$, ! 11 (\forall E: C1.14) i

$\theta (1 \ \square \ b) \ \& \ \leq[1, 1]$, ! 12 (&I: V3.49, 11) i

$\leq[1, (\lambda F)]$, ! 13 (=E: V3.49, 5) i

$\theta (1 \ \square \ b) \ \& \ \leq[1, 1] \ \& \ \leq[1, (\lambda F)]$, ! 14 (&I: 12, 13) i

$(\theta (1 \ \square \ b) \ \& \ \leq[1, 1] \ \& \ \leq[1, (\lambda F)]) \Rightarrow ((F^{(1 \ \square \ b)})^{(1)}) = (F'1)$
, ! 15 (\forall E: P16) i

$\theta (1 \ \square \ b) \ \& \ \leq[1, 1] \ \& \ \leq[1, (\lambda F)] \Rightarrow ((F^{(1 \ \square \ b)})^{(1)}) = (F'1)$
, ! 16 (()E: 15) i

$((\mathbf{F}^{\wedge}(1 \ \blacksquare \ \mathbf{b}))'1) = (\mathbf{F}'1)$,! 17 (\Rightarrow E: 14,16) ;
 $((\mathbf{F}^{\wedge}(1 \ \blacksquare \ \mathbf{b}))'1) = \mathbf{a}$,! 18 (=E: 6,17) ;
 $(\lambda(\mathbf{F}^{\wedge}(1 \ \blacksquare \ \mathbf{b}))) = 2 \ \& \ ((\mathbf{F}^{\wedge}(1 \ \blacksquare \ \mathbf{b}))'1) = \mathbf{a} \ \& \ ((\mathbf{F}^{\wedge}(1 \ \blacksquare \ \mathbf{b}))'2) = \mathbf{b}$
, ! 19 (&I: 10,18) ;
 $((\lambda(\mathbf{F}^{\wedge}(1 \ \blacksquare \ \mathbf{b}))) = 2 \ \& \ ((\mathbf{F}^{\wedge}(1 \ \blacksquare \ \mathbf{b}))'1) = \mathbf{a}$
 $\ \& \ ((\mathbf{F}^{\wedge}(1 \ \blacksquare \ \mathbf{b}))'2) = \mathbf{b})$
, ! 20 (()I: 19) ;
 $\exists \mathbf{F} ((\lambda \mathbf{F}) = 2 \ \& \ (\mathbf{F}'1) = \mathbf{a} \ \& \ (\mathbf{F}'2) = \mathbf{b})$,! 21 (\exists I: 20) ;
 $\forall \mathbf{a} \forall \mathbf{b} \exists \mathbf{F} ((\lambda \mathbf{F}) = 2 \ \& \ (\mathbf{F}'1) = \mathbf{a} \ \& \ (\mathbf{F}'2) = \mathbf{b})$
! 22 (\forall I: 1,21) ;
 \square
! 39. ;
 $\vdash \forall \mathbf{a} \forall \mathbf{b} \forall \mathbf{c} \exists \mathbf{F} ((\lambda \mathbf{F}) = 3 \ \& \ (\mathbf{F}'1) = \mathbf{a} \ \& \ (\mathbf{F}'2) = \mathbf{b} \ \& \ (\mathbf{F}'3) = \mathbf{c})$;
 $\mathbf{a}, \mathbf{b}, \mathbf{c}$,! 1 (Prem) ;
 $\exists \mathbf{F} ((\lambda \mathbf{F}) = 2 \ \& \ (\mathbf{F}'1) = \mathbf{a} \ \& \ (\mathbf{F}'2) = \mathbf{b})$,! 2 (\forall E: P38) ;
 $((\lambda \mathbf{F}) = 2 \ \& \ (\mathbf{F}'1) = \mathbf{a} \ \& \ (\mathbf{F}'2) = \mathbf{b})$,! 3 (\exists E: 2) ;
 $(\lambda \mathbf{F}) = 2 \ \& \ (\mathbf{F}'1) = \mathbf{a} \ \& \ (\mathbf{F}'2) = \mathbf{b}$,! 4 (()E: 3) ;
 $(\lambda \mathbf{F}) = 2$,! 5 (&E: 4) ;
 $(\mathbf{F}'1) = \mathbf{a}$,! 6 (&E: 4) ;
 $(\mathbf{F}'2) = \mathbf{b}$,! 7 (&E: 4) ;
 $((\lambda \mathbf{F}) = 2$
 $\Rightarrow (\lambda(\mathbf{F}^{\wedge}(1 \ \blacksquare \ \mathbf{c}))) = (2+1) \ \& \ ((\mathbf{F}^{\wedge}(1 \ \blacksquare \ \mathbf{c}))'(2+1)) = \mathbf{c})$
, ! 8 (\forall E: P29) ;
 $(\lambda \mathbf{F}) = 2$
 $\Rightarrow (\lambda(\mathbf{F}^{\wedge}(1 \ \blacksquare \ \mathbf{c}))) = (2+1) \ \& \ ((\mathbf{F}^{\wedge}(1 \ \blacksquare \ \mathbf{c}))'(2+1)) = \mathbf{c}$
, ! 9 (()E: 8) ;
 $(\lambda(\mathbf{F}^{\wedge}(1 \ \blacksquare \ \mathbf{c}))) = (2+1) \ \& \ ((\mathbf{F}^{\wedge}(1 \ \blacksquare \ \mathbf{c}))'(2+1)) = \mathbf{c}$
, ! 10 (\Rightarrow E: 5,9) ;
 $(\lambda(\mathbf{F}^{\wedge}(1 \ \blacksquare \ \mathbf{c}))) = 3 \ \& \ ((\mathbf{F}^{\wedge}(1 \ \blacksquare \ \mathbf{c}))'3) = \mathbf{c}$
, ! 11 (=E: V2.70,10) ;
 $\theta (1 \ \blacksquare \ \mathbf{c})$,! 12 (\forall E: C1.14) ;
 $\theta (1 \ \blacksquare \ \mathbf{c}) \ \& \ \leq[1,1]$,! 13 (&I: V3.49,12) ;

$\leq[1, (\lambda F)]$,! 14 (=E: V3.50,5) ;
 $\theta (1 \dashv \vdash c) \ \& \ \leq[1,1] \ \& \ \leq[1, (\lambda F)]$,! 15 (&I: 13,14) ;
 $(\theta (1 \dashv \vdash c) \ \& \ \leq[1,1] \ \& \ \leq[1, (\lambda F)] \Rightarrow ((F^{\wedge}(1 \dashv \vdash c))^{\prime}1) = (F^{\prime}1))$
, ! 16 (\forall E: P16) ;
 $\theta (1 \dashv \vdash c) \ \& \ \leq[1,1] \ \& \ \leq[1, (\lambda F)] \Rightarrow ((F^{\wedge}(1 \dashv \vdash c))^{\prime}1) = (F^{\prime}1)$
, ! 17 (()E: 16) ;
 $((F^{\wedge}(1 \dashv \vdash c))^{\prime}1) = (F^{\prime}1)$,! 18 (\Rightarrow E: 15,17) ;
 $((F^{\wedge}(1 \dashv \vdash c))^{\prime}1) = a$,! 19 (=E: 6,18) ;
 $(\lambda(F^{\wedge}(1 \dashv \vdash c))) = 3 \ \& \ ((F^{\wedge}(1 \dashv \vdash c))^{\prime}1) = a \ \& \ ((F^{\wedge}(1 \dashv \vdash c))^{\prime}3) = c$
, ! 20 (&I: 11,19) ;
 $\theta (1 \dashv \vdash c) \ \& \ \leq[1,2]$,! 21 (&I: V3.50,12) ;
 $\leq[2, (\lambda F)]$,! 22 (=E: V3.51,5) ;
 $\theta (1 \dashv \vdash c) \ \& \ \leq[1,2] \ \& \ \leq[2, (\lambda F)]$,! 23 (&I: 21,22) ;
 $(\theta (1 \dashv \vdash c) \ \& \ \leq[1,2] \ \& \ \leq[2, (\lambda F)] \Rightarrow ((F^{\wedge}(1 \dashv \vdash c))^{\prime}2) = (F^{\prime}2))$
, ! 24 (\forall E: P16) ;
 $\theta (1 \dashv \vdash c) \ \& \ \leq[1,2] \ \& \ \leq[2, (\lambda F)] \Rightarrow ((F^{\wedge}(1 \dashv \vdash c))^{\prime}2) = (F^{\prime}2)$
, ! 25 (()E: 24) ;
 $((F^{\wedge}(1 \dashv \vdash c))^{\prime}2) = (F^{\prime}2)$,! 26 (\Rightarrow E: 23,25) ;
 $((F^{\wedge}(1 \dashv \vdash c))^{\prime}2) = b$,! 27 (=E: 7,26) ;
 $(\lambda(F^{\wedge}(1 \dashv \vdash c))) = 3 \ \& \ ((F^{\wedge}(1 \dashv \vdash c))^{\prime}1) = a$
 $\ \& \ ((F^{\wedge}(1 \dashv \vdash c))^{\prime}2) = b \ \& \ ((F^{\wedge}(1 \dashv \vdash c))^{\prime}3) = c$
, ! 28 (&I: 20,27) ;
 $((\lambda(F^{\wedge}(1 \dashv \vdash c))) = 3 \ \& \ ((F^{\wedge}(1 \dashv \vdash c))^{\prime}1) = a$
 $\ \& \ ((F^{\wedge}(1 \dashv \vdash c))^{\prime}2) = b \ \& \ ((F^{\wedge}(1 \dashv \vdash c))^{\prime}3) = c)$
, ! 29 (()I: 28) ;
 $\exists F ((\lambda F) = 3 \ \& \ (F^{\prime}1) = a \ \& \ (F^{\prime}2) = b \ \& \ (F^{\prime}3) = c)$
, ! 30 (\exists I: 29) ;
 $\forall a \forall b \forall c \exists F ((\lambda F) = 3 \ \& \ (F^{\prime}1) = a \ \& \ (F^{\prime}2) = b \ \& \ (F^{\prime}3) = c)$
! 31 (\forall I: 1,30) ;

□