

## ! CHAPTER 2

### BASIC LAWS OF ADDITION;

! This chapter establishes most of the basic laws of addition.  
Highlights include:

P5: Commutative Law of Addition

P14: Associative Law of Addition

P30-P33: 0 is Additive Identity

P36-P39: Uniqueness of 0 as Additive Identity

P60-P63: Cancellation Laws of Addition

P69:  $(1 + 1) = 2$

P74:  $(2 + 2) = 4$

i

! P1 through P4 substitute equals for equals in an addition term.

i

! 1.

i

$\vdash \forall n \forall m \forall k ( \omega[n] \ \& \ \omega[k] \ \& \ n = m \Rightarrow (n + k) = (m + k) )$

i

**n, m, k**

,! 1 (Prem)

i

$\omega[n] \ \& \ \omega[k] \ \& \ n = m$

,! 2 (Prem)

i

$\omega[n] \ \& \ \omega[k]$

,! 3 (&E: 2)

i

$n = m$

,! 4 (&E: 2)

i

$(n + k) = (n + k)$

,! 5 (=I;

$(n + k)$ : C1.7,3)

i

$(n + k) = (m + k)$

,! 6 (=E: 4,5)

i

$\omega[n] \ \& \ \omega[k] \ \& \ n = m \Rightarrow (n + k) = (m + k)$

,! 7 ( $\Rightarrow$ I: 2,6)

i

$( \omega[n] \ \& \ \omega[k] \ \& \ n = m \Rightarrow (n + k) = (m + k) )$

,! 8 (( )I: 7)

i

$\forall n \forall m \forall k ( \omega[n] \ \& \ \omega[k] \ \& \ n = m \Rightarrow (n + k) = (m + k) )$

! 9 ( $\forall$ I: 1,8)

i

$\square$

! 2.

i

$\vdash \forall n \forall m \forall k ( \omega[n] \ \& \ \omega[k] \ \& \ n = m \Rightarrow (k + n) = (k + m) )$

i

**n, m, k**

,! 1 (Prem)

i

$\omega[n] \ \& \ \omega[k] \ \& \ n = m$

,! 2 (Prem)

i

$\omega[n]$

,! 3 (&E: 2)

i

$\omega[k]$

,! 4 (&E: 2)

i

$n = m$

,! 5 (&E: 2)

i

$\omega[\mathbf{k}] \ \& \ \omega[\mathbf{n}]$	,! 6 (&I: 3,4)	i
$(\mathbf{k} + \mathbf{n}) = (\mathbf{k} + \mathbf{n})$	,! 7 (=I; $(\mathbf{k} + \mathbf{n}): C1.7,6)$	i
$(\mathbf{k} + \mathbf{n}) = (\mathbf{k} + \mathbf{m})$	,! 8 (=E: 5,7)	i
$\omega[\mathbf{n}] \ \& \ \omega[\mathbf{k}] \ \& \ \mathbf{n} = \mathbf{m} \Rightarrow (\mathbf{k} + \mathbf{n}) = (\mathbf{k} + \mathbf{m})$	,! 9 ( $\Rightarrow$ I: 2,8)	i
$( \ \omega[\mathbf{n}] \ \& \ \omega[\mathbf{k}] \ \& \ \mathbf{n} = \mathbf{m} \Rightarrow (\mathbf{k} + \mathbf{n}) = (\mathbf{k} + \mathbf{m}) \ )$	,! 10 (( )I: 9)	i
$\forall n \forall m \forall k ( \ \omega[n] \ \& \ \omega[k] \ \& \ n = m \Rightarrow (k + n) = (k + m) \ )$	! 11 ( $\forall$ I: 1,10)	i
$\square$		
<b>! 3.</b>		i
$\vdash \forall n \forall m \forall k \forall j ( \ \omega[j] \ \& \ (n + m) = k \Rightarrow ((n + m) + j) = (k + j) \ )$		i
$\mathbf{n}, \mathbf{m}, \mathbf{k}, \mathbf{j}$	,! 1 (Prem)	i
$\omega[\mathbf{j}] \ \& \ (\mathbf{n} + \mathbf{m}) = \mathbf{k}$	,! 2 (Prem)	i
$\omega[\mathbf{j}]$	,! 3 (&E: 2)	i
$(\mathbf{n} + \mathbf{m}) = \mathbf{k}$	,! 4 (&E: 2)	i
$\omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}]$	,! 5 ( $\mathbb{T}$ E: C1.7,4)	i
$( \ \omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}] \Rightarrow \omega[(\mathbf{n} + \mathbf{m})] \ )$	,! 6 ( $\forall$ E: C1.8)	i
$\omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}] \Rightarrow \omega[(\mathbf{n} + \mathbf{m})]$	,! 7 (( )E: 6)	i
$\omega[(\mathbf{n} + \mathbf{m})]$	,! 8 ( $\Rightarrow$ E: 5,7)	i
$\omega[(\mathbf{n} + \mathbf{m})] \ \& \ \omega[\mathbf{j}]$	,! 9 (&I: 3,8)	i
$((\mathbf{n} + \mathbf{m}) + \mathbf{j}) = ((\mathbf{n} + \mathbf{m}) + \mathbf{j})$	,! 10 (=I; $((\mathbf{n} + \mathbf{m}) + \mathbf{j}): C1.7,9)$	i
$((\mathbf{n} + \mathbf{m}) + \mathbf{j}) = (\mathbf{k} + \mathbf{j})$	,! 11 (=E: 4,10)	i
$\omega[\mathbf{j}] \ \& \ (\mathbf{n} + \mathbf{m}) = \mathbf{k} \Rightarrow ((\mathbf{n} + \mathbf{m}) + \mathbf{j}) = (\mathbf{k} + \mathbf{j})$	,! 12 ( $\Rightarrow$ I: 2,11)	i
$( \ \omega[\mathbf{j}] \ \& \ (\mathbf{n} + \mathbf{m}) = \mathbf{k} \Rightarrow ((\mathbf{n} + \mathbf{m}) + \mathbf{j}) = (\mathbf{k} + \mathbf{j}) \ )$	,! 13 (( )I: 12)	i
$\forall n \forall m \forall k \forall j ( \ \omega[j] \ \& \ (n + m) = k \Rightarrow ((n + m) + j) = (k + j) \ )$	! 14 ( $\forall$ I: 1,13)	i
$\square$		

! 4. i

$\vdash \forall n \forall m \forall k \forall j ( \omega[j] \ \& \ (n + m) = k \Rightarrow (j + (n + m)) = (j + k) )$  ;

**n, m, k, j** ,! 1 (Prem) i

$\omega[j] \ \& \ (n + m) = k$  ,! 2 (Prem) i

$\omega[j]$  ,! 3 (&E: 2) i

$(n + m) = k$  ,! 4 (&E: 2) i

$\omega[n] \ \& \ \omega[m]$  ,! 5 (**TE**: C1.7,4) i

$( \omega[n] \ \& \ \omega[m] \Rightarrow \omega[(n + m)] )$  ,! 6 ( $\forall$ E: C1.8) i

$\omega[n] \ \& \ \omega[m] \Rightarrow \omega[(n + m)]$  ,! 7 (( )E: 6) i

$\omega[(n + m)]$  ,! 8 ( $\Rightarrow$ E: 5,7) i

$\omega[j] \ \& \ \omega[(n + m)]$  ,! 9 (&I: 3,8) i

$(j + (n + m)) = (j + (n + m))$  ,! 10 (=I;  
(j + (n + m)): C1.7,9) i

$(j + (n + m)) = (j + k)$  ,! 11 (=E: 4,10) i

$\omega[j] \ \& \ (n + m) = k \Rightarrow (j + (n + m)) = (j + k)$  ,! 12 ( $\Rightarrow$ I: 2,11) i

$( \omega[j] \ \& \ (n + m) = k \Rightarrow (j + (n + m)) = (j + k) )$  ,! 13 (( )I: 12) i

$\forall n \forall m \forall k \forall j ( \omega[j] \ \& \ (n + m) = k \Rightarrow (j + (n + m)) = (j + k) )$  ! 14 ( $\forall$ I: 1,13) i

□

! 5. The Commutative Law of Addition. i

$\vdash \forall n \forall m ( \omega[n] \ \& \ \omega[m] \Rightarrow (n + m) = (m + n) )$  ;

**n, m** ,! 1 (Prem) i

$\omega[n] \ \& \ \omega[m]$  ,! 2 (Prem) i

$( \omega[n] \ \& \ \omega[m] \Rightarrow \omega[(n+m)] )$  ,! 3 ( $\forall$ E: C1.8) i

$\omega[n] \ \& \ \omega[m] \Rightarrow \omega[(n+m)]$  ,! 4 (( )E: 3) i

$\omega[(n+m)]$  ,! 5 ( $\Rightarrow$ E: 2,4) i

$( \omega[n] \ \& \ \omega[m] \Rightarrow \exists A \exists B ( \mathcal{N}_k[n, A] \ \& \ \mathcal{N}_k[m, B] \ \& \ (A \cap B) \equiv \phi )$

$$\begin{aligned} & \& \mathcal{N}[(n+m), (A \cup B)] \text{ )} \\ & \text{, ! 6 (}\forall\text{E: C1.18) } \quad \text{i} \\ \\ \omega[n] \& \omega[m] \\ \Rightarrow \exists A \exists B \text{ (} \mathcal{N}[n, A] \& \mathcal{N}[m, B] \& (A \cap B) \equiv \phi \& \mathcal{N}[(n+m), (A \cup B)] \text{ )} \\ & \text{, ! 7 (}\text{()E: 6) } \quad \text{i} \\ \\ \exists A \exists B \text{ (} \mathcal{N}[n, A] \& \mathcal{N}[m, B] \& (A \cap B) \equiv \phi \& \mathcal{N}[(n+m), (A \cup B)] \text{ )} \\ & \text{, ! 8 (}\Rightarrow\text{E: 2,7) } \quad \text{i} \\ \\ \exists B \text{ (} \mathcal{N}[n, A] \& \mathcal{N}[m, B] \& (A \cap B) \equiv \phi \& \mathcal{N}[(n+m), (A \cup B)] \text{ )} \\ & \text{, ! 9 (}\exists\text{E: 8) } \quad \text{i} \\ \\ \text{(} \mathcal{N}[n, A] \& \mathcal{N}[m, B] \& (A \cap B) \equiv \phi \& \mathcal{N}[(n+m), (A \cup B)] \text{ )} \\ & \text{, ! 10 (}\exists\text{E: 9) } \quad \text{i} \\ \\ \mathcal{N}[n, A] \& \mathcal{N}[m, B] \& (A \cap B) \equiv \phi \& \mathcal{N}[(n+m), (A \cup B)] \\ & \text{, ! 11 (}\text{()E: 10) } \quad \text{i} \\ \\ \mathcal{N}[(n+m), (A \cup B)] \\ & \text{, ! 12 (}\&\text{E: 11) } \quad \text{i} \\ \\ \omega[(n+m)] \& \mathcal{N}[(n+m), (A \cup B)] \\ & \text{, ! 13 (}\&\text{I: 5,12) } \quad \text{i} \\ \\ (A \cup B) \equiv (B \cup A) \\ & \text{, ! 14 (}\forall\text{E: II2.16) } \quad \text{i} \\ \\ \omega[(n+m)] \& \mathcal{N}[(n+m), (A \cup B)] \& (A \cup B) \equiv (B \cup A) \\ & \text{, ! 15 (}\&\text{I: 13,14) } \quad \text{i} \\ \\ (\omega[(n+m)] \& \mathcal{N}[(n+m), (A \cup B)] \& (A \cup B) \equiv (B \cup A) \\ \Rightarrow \mathcal{N}[(n+m), (B \cup A)] \text{ )} \\ & \text{, ! 16 (}\forall\text{E: IV4.5; } \\ & \text{(n+m): C1.7,2) } \quad \text{i} \\ \\ \omega[(n+m)] \& \mathcal{N}[(n+m), (A \cup B)] \& (A \cup B) \equiv (B \cup A) \\ \Rightarrow \mathcal{N}[(n+m), (B \cup A)] \\ & \text{, ! 17 (}\text{()E: 16) } \quad \text{i} \\ \\ \mathcal{N}[(n+m), (B \cup A)] \\ & \text{, ! 18 (}\Rightarrow\text{E: 15,17) } \quad \text{i} \\ \\ \omega[n] \\ & \text{, ! 19 (}\&\text{E: 2) } \quad \text{i} \\ \\ \omega[m] \\ & \text{, ! 20 (}\&\text{E: 2) } \quad \text{i} \\ \\ \mathcal{N}[n, A] \\ & \text{, ! 21 (}\&\text{E: 11) } \quad \text{i} \\ \\ \mathcal{N}[m, B] \\ & \text{, ! 22 (}\&\text{E: 11) } \quad \text{i} \\ \\ \omega[m] \& \omega[n] \\ & \text{, ! 23 (}\&\text{I: 19,20) } \quad \text{i} \\ \\ \omega[m] \& \omega[n] \& \omega[(n+m)] \\ & \text{, ! 24 (}\&\text{I: 5,23) } \quad \text{i} \\ \\ \omega[m] \& \omega[n] \& \omega[(n+m)] \& \mathcal{N}[m, B] \\ & \text{, ! 25 (}\&\text{I: 22,24) } \quad \text{i} \end{aligned}$$

$\omega[m] \ \& \ \omega[n] \ \& \ \omega[(n+m)] \ \& \ \mathcal{N}[m,B] \ \& \ \mathcal{N}[n,A]$	,! 26 (&I: 21,25)	i
$(A \cap B) \equiv \phi$	,! 27 (&E: 11)	i
$( (A \cap B) \equiv \phi \Rightarrow (B \cap A) \equiv \phi )$	,! 28 ( $\forall$ E: II3.17)	i
$(A \cap B) \equiv \phi \Rightarrow (B \cap A) \equiv \phi$	,! 29 (( )E: 28)	i
$(B \cap A) \equiv \phi$	,! 30 ( $\Rightarrow$ E: 27,29)	i
$\omega[m] \ \& \ \omega[n] \ \& \ \omega[(n+m)] \ \& \ \mathcal{N}[m,B] \ \& \ \mathcal{N}[n,A] \ \& \ (B \cap A) \equiv \phi$	,! 31 (&I: 26,30)	i
$\omega[m] \ \& \ \omega[n] \ \& \ \omega[(n+m)] \ \& \ \mathcal{N}[m,B] \ \& \ \mathcal{N}[n,A] \ \& \ (B \cap A) \equiv \phi$ $\ \& \ \mathcal{N}[(n+m), (B \cup A)]$	,! 32 (&I: 18,31)	i
$( \omega[m] \ \& \ \omega[n] \ \& \ \omega[(n+m)] \ \& \ \mathcal{N}[m,B] \ \& \ \mathcal{N}[n,A] \ \& \ (B \cap A) \equiv \phi$ $\ \& \ \mathcal{N}[(n+m), (B \cup A)]$ $\Rightarrow (n+m) = (m + n) )$	,! 33 ( $\forall$ E: C1.17; (n+m): C1.7,2)	i
$\omega[m] \ \& \ \omega[n] \ \& \ \omega[(n+m)] \ \& \ \mathcal{N}[m,B] \ \& \ \mathcal{N}[n,A] \ \& \ (B \cap A) \equiv \phi$ $\ \& \ \mathcal{N}[(n+m), (B \cup A)]$ $\Rightarrow (n+m) = (m + n)$	,! 34 (( )E: 33)	i
$(n + m) = (m + n)$	,! 35 ( $\Rightarrow$ E: 32,34)	i
$\omega[n] \ \& \ \omega[m] \Rightarrow (n + m) = (m + n)$	,! 36 ( $\Rightarrow$ I: 2,35)	i
$( \omega[n] \ \& \ \omega[m] \Rightarrow (n + m) = (m + n) )$	,! 37 (( )I: 36)	i
$\forall n \forall m ( \omega[n] \ \& \ \omega[m] \Rightarrow (n + m) = (m + n) )$	! 38 ( $\forall$ I: 1,37)	i

□

! P6 and P7 are derivative forms of the Commutative Law. i

! 6. i

$\vdash \forall n \forall m \forall k ( (n + m) = k \Rightarrow (m + n) = k )$  i

$n, m, k$  ,! 1 (Prem) i

$(n + m) = k$  ,! 2 (Prem) i

$\omega[n] \ \& \ \omega[m]$  ,! 3 ( $\mathbb{T}$ E: C1.7,2) i

$( \omega[n] \ \& \ \omega[m] \Rightarrow (n + m) = (m + n) )$  ,! 4 ( $\forall$ E: P5) i

$\omega[n] \ \& \ \omega[m] \Rightarrow (n + m) = (m + n)$  ,! 5 (( )E: 4) i

$(n + m) = (m + n)$	,! 6 ( $\Rightarrow$ E: 3,5)	i
$(m + n) = k$	,! 7 (=E: 2,6)	i
$(n + m) = k \Rightarrow (m + n) = k$	,! 8 ( $\Rightarrow$ I: 2,7)	i
$( (n + m) = k \Rightarrow (m + n) = k )$	,! 9 (( )I: 8)	i
$\forall n \forall m \forall k ( (n + m) = k \Rightarrow (m + n) = k )$	! 10 ( $\forall$ I: 1,9)	i

□

! 7. i

$\vdash \forall n \forall m \forall k ( k = (n + m) \Rightarrow k = (m + n) )$	i
<b>n, m, k</b>	,! 1 (Prem) i
$k = (n + m)$	,! 2 (Prem) i
$\omega[n] \ \& \ \omega[m]$	,! 3 ( $\mathbb{T}$ E: C1.7,2) i
$( \omega[n] \ \& \ \omega[m] \Rightarrow (n + m) = (m + n) )$	,! 4 ( $\forall$ E: P5) i
$\omega[n] \ \& \ \omega[m] \Rightarrow (n + m) = (m + n)$	,! 5 (( )E: 4) i
$(n + m) = (m + n)$	,! 6 ( $\Rightarrow$ E: 3,5) i
$k = (m + n)$	,! 7 (=E: 2,6) i
$k = (n + m) \Rightarrow k = (m + n)$	,! 8 ( $\Rightarrow$ I: 2,7) i
$( k = (n + m) \Rightarrow k = (m + n) )$	,! 9 (( )I: 8) i
$\forall n \forall m \forall k ( k = (n + m) \Rightarrow k = (m + n) )$	! 10 ( $\forall$ I: 1,9) i

□

! P8 through P13 are easy applications of the Commutative Law. i

! 8. i

$\vdash \forall n \forall m \forall k ( \omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow ((n + m) + k) = ((m + n) + k) )$	i
<b>n, m, k</b>	,! 1 (Prem) i
$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$	,! 2 (Prem) i
$\omega[n] \ \& \ \omega[m]$	,! 3 ( $\&$ E: 2) i
$\omega[k]$	,! 4 ( $\&$ E: 2) i
$( \omega[n] \ \& \ \omega[m] \Rightarrow \omega[(n + m)] )$	,! 5 ( $\forall$ E: C1.8) i

$\omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}] \Rightarrow \omega[(\mathbf{n} + \mathbf{m})]$	, ! 6 ( )E: 5	i
$\omega[(\mathbf{n} + \mathbf{m})]$	, ! 7 ( $\Rightarrow$ E: 3,6)	i
$\omega[(\mathbf{n} + \mathbf{m})] \ \& \ \omega[\mathbf{k}]$	, ! 8 (&I: 4,7)	i
$((\mathbf{n} + \mathbf{m}) + \mathbf{k}) = ((\mathbf{n} + \mathbf{m}) + \mathbf{k})$	, ! 9 (=I; $((\mathbf{n} + \mathbf{m}) + \mathbf{k})$ : C1.7,8)	i
$(\omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}] \Rightarrow (\mathbf{n} + \mathbf{m}) = (\mathbf{m} + \mathbf{n}))$	, ! 10 ( $\forall$ E: P5)	i
$\omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}] \Rightarrow (\mathbf{n} + \mathbf{m}) = (\mathbf{m} + \mathbf{n})$	, ! 11 ( )E: 10	i
$(\mathbf{n} + \mathbf{m}) = (\mathbf{m} + \mathbf{n})$	, ! 12 ( $\Rightarrow$ E: 3,11)	i
$((\mathbf{n} + \mathbf{m}) + \mathbf{k}) = ((\mathbf{m} + \mathbf{n}) + \mathbf{k})$	, ! 13 (=E: 9,12)	i
$\omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}] \ \& \ \omega[\mathbf{k}] \Rightarrow ((\mathbf{n} + \mathbf{m}) + \mathbf{k}) = ((\mathbf{m} + \mathbf{n}) + \mathbf{k})$	, ! 14 ( $\Rightarrow$ I: 2,13)	i
$(\omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}] \ \& \ \omega[\mathbf{k}] \Rightarrow ((\mathbf{n} + \mathbf{m}) + \mathbf{k}) = ((\mathbf{m} + \mathbf{n}) + \mathbf{k}))$	, ! 15 ( )I: 14	i
$\forall \mathbf{n} \forall \mathbf{m} \forall \mathbf{k} (\omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}] \ \& \ \omega[\mathbf{k}] \Rightarrow ((\mathbf{n} + \mathbf{m}) + \mathbf{k}) = ((\mathbf{m} + \mathbf{n}) + \mathbf{k}))$	! 16 ( $\forall$ I: 1,15)	i

□

! 9.

$\vdash \forall \mathbf{n} \forall \mathbf{m} \forall \mathbf{k} \forall \mathbf{j} ((\mathbf{n} + \mathbf{m}) + \mathbf{k}) = \mathbf{j} \Rightarrow ((\mathbf{m} + \mathbf{n}) + \mathbf{k}) = \mathbf{j}$	i	
$\mathbf{n}, \mathbf{m}, \mathbf{k}, \mathbf{j}$	, ! 1 (Prem)	i
$((\mathbf{n} + \mathbf{m}) + \mathbf{k}) = \mathbf{j}$	, ! 2 (Prem)	i
$\omega[(\mathbf{n} + \mathbf{m})] \ \& \ \omega[\mathbf{k}]$	, ! 3 ( $\mathbb{T}$ E: C1.7,2)	i
$\omega[(\mathbf{n} + \mathbf{m})]$	, ! 4 (&E: 3)	i
$\omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}]$	, ! 5 ( $\mathbb{T}$ E: C1.7,4)	i
$\omega[\mathbf{k}]$	, ! 6 (&E: 3)	i
$\omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}] \ \& \ \omega[\mathbf{k}]$	, ! 7 (&I: 5,6)	i
$(\omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}] \ \& \ \omega[\mathbf{k}] \Rightarrow ((\mathbf{n} + \mathbf{m}) + \mathbf{k}) = ((\mathbf{m} + \mathbf{n}) + \mathbf{k}))$	, ! 8 ( $\forall$ E: P8)	i
$\omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}] \ \& \ \omega[\mathbf{k}] \Rightarrow ((\mathbf{n} + \mathbf{m}) + \mathbf{k}) = ((\mathbf{m} + \mathbf{n}) + \mathbf{k})$	, ! 9 ( )E: 8	i
$((\mathbf{n} + \mathbf{m}) + \mathbf{k}) = ((\mathbf{m} + \mathbf{n}) + \mathbf{k})$	, ! 10 ( $\Rightarrow$ E: 7,9)	i
$((\mathbf{m} + \mathbf{n}) + \mathbf{k}) = \mathbf{j}$	, ! 11 (=E: 2,10)	i

$((n + m) + k) = j \Rightarrow ((m + n) + k) = j$  ,! 12 ( $\Rightarrow$ I: 2,11) i

(  $((n + m) + k) = j \Rightarrow ((m + n) + k) = j$  )  
,! 13 ( $()$ I: 12) i

$\forall n \forall m \forall k \forall j$  (  $((n + m) + k) = j \Rightarrow ((m + n) + k) = j$  )  
! 14 ( $\forall$ I: 1,13) i

□

! 10. i

$\vdash \forall n \forall m \forall k \forall j$  (  $j = ((n + m) + k) \Rightarrow j = ((m + n) + k)$  ) i

**n,m,k,j** ,! 1 (Prem) i

**j = ((n + m) + k)** ,! 2 (Prem) i

**j = j** ,! 3 (=I) i

**((n + m) + k) = j** ,! 4 (=E: 2,3) i

(  $((n + m) + k) = j \Rightarrow ((m + n) + k) = j$  )  
,! 5 ( $\forall$ E: P9) i

**((n + m) + k) = j  $\Rightarrow$  ((m + n) + k) = j**  
,! 6 ( $()$ E: 5) i

**((m + n) + k) = j** ,! 7 ( $\Rightarrow$ E: 4,6) i

**j = ((m + n) + k)** ,! 8 (=E: 3,7) i

**j = ((n + m) + k)  $\Rightarrow$  j = ((m + n) + k)** ,! 9 ( $\Rightarrow$ I: 2,8) i

( **j = ((n + m) + k)  $\Rightarrow$  j = ((m + n) + k)** )  
,! 10 ( $()$ I: 9) i

$\forall n \forall m \forall k \forall j$  ( **j = ((n + m) + k)  $\Rightarrow$  j = ((m + n) + k)** )  
! 11 ( $\forall$ I: 1,10) i

□

! 11. i

$\vdash \forall n \forall m \forall k$  (  $\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow (n + (m + k)) = (n + (k + m))$  ) i

**n,m,k** ,! 1 (Prem) i

**$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$**  ,! 2 (Prem) i

**$\omega[n]$**  ,! 3 ( $\&$ E: 2) i

**$\omega[m] \ \& \ \omega[k]$**  ,! 4 ( $\&$ E: 2) i

(  **$\omega[m] \ \& \ \omega[k] \Rightarrow \omega[(m + k)]$**  ) ,! 5 ( $\forall$ E: C1.8) i

$\omega[\mathbf{m}] \ \& \ \omega[\mathbf{k}] \Rightarrow \omega[(\mathbf{m} + \mathbf{k})]$	,! 6 ((E: 5)	i
$\omega[(\mathbf{m} + \mathbf{k})]$	,! 7 ( $\Rightarrow$ E: 4,6)	i
$\omega[\mathbf{n}] \ \& \ \omega[(\mathbf{m} + \mathbf{k})]$	,! 8 (&I: 3,7)	i
$(\mathbf{n} + (\mathbf{m} + \mathbf{k})) = (\mathbf{n} + (\mathbf{m} + \mathbf{k}))$	,! 9 (=I; ( $\mathbf{n} + (\mathbf{m} + \mathbf{k}))$ ): C1.7,8)	i
$(\omega[\mathbf{m}] \ \& \ \omega[\mathbf{k}] \Rightarrow (\mathbf{m} + \mathbf{k}) = (\mathbf{k} + \mathbf{m}))$	,! 10 ( $\forall$ E: P5)	i
$\omega[\mathbf{m}] \ \& \ \omega[\mathbf{k}] \Rightarrow (\mathbf{m} + \mathbf{k}) = (\mathbf{k} + \mathbf{m})$	,! 11 ((E: 10)	i
$(\mathbf{m} + \mathbf{k}) = (\mathbf{k} + \mathbf{m})$	,! 12 ( $\Rightarrow$ E: 4,11)	i
$(\mathbf{n} + (\mathbf{m} + \mathbf{k})) = (\mathbf{n} + (\mathbf{k} + \mathbf{m}))$	,! 13 (=E: 9,12)	i
$\omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}] \ \& \ \omega[\mathbf{k}] \Rightarrow (\mathbf{n} + (\mathbf{m} + \mathbf{k})) = (\mathbf{n} + (\mathbf{k} + \mathbf{m}))$	,! 14 ( $\Rightarrow$ I: 2,13)	i
$(\omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}] \ \& \ \omega[\mathbf{k}] \Rightarrow (\mathbf{n} + (\mathbf{m} + \mathbf{k})) = (\mathbf{n} + (\mathbf{k} + \mathbf{m})))$	,! 15 ((I: 14)	i
$\forall n \forall m \forall k (\omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}] \ \& \ \omega[\mathbf{k}] \Rightarrow (\mathbf{n} + (\mathbf{m} + \mathbf{k})) = (\mathbf{n} + (\mathbf{k} + \mathbf{m})))$	! 16 ( $\forall$ I: 1,15)	i

□

! 12.

$\vdash \forall n \forall m \forall k \forall j ((\mathbf{n} + (\mathbf{m} + \mathbf{k})) = \mathbf{j} \Rightarrow (\mathbf{n} + (\mathbf{k} + \mathbf{m})) = \mathbf{j})$	i	
$\mathbf{n}, \mathbf{m}, \mathbf{k}, \mathbf{j}$	,! 1 (Prem)	i
$(\mathbf{n} + (\mathbf{m} + \mathbf{k})) = \mathbf{j}$	,! 2 (Prem)	i
$\omega[\mathbf{n}] \ \& \ \omega[(\mathbf{m} + \mathbf{k})]$	,! 3 ( $\mathbb{T}$ E: C1.7,2)	i
$\omega[\mathbf{n}]$	,! 4 (&E: 3)	i
$\omega[(\mathbf{m} + \mathbf{k})]$	,! 5 (&E: 3)	i
$\omega[\mathbf{m}] \ \& \ \omega[\mathbf{k}]$	,! 6 ( $\mathbb{T}$ E: C1.7,5)	i
$\omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}] \ \& \ \omega[\mathbf{k}]$	,! 7 (&I: 4,6)	i
$(\omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}] \ \& \ \omega[\mathbf{k}] \Rightarrow (\mathbf{n} + (\mathbf{m} + \mathbf{k})) = (\mathbf{n} + (\mathbf{k} + \mathbf{m})))$	,! 8 ( $\forall$ E: P11)	i
$\omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}] \ \& \ \omega[\mathbf{k}] \Rightarrow (\mathbf{n} + (\mathbf{m} + \mathbf{k})) = (\mathbf{n} + (\mathbf{k} + \mathbf{m}))$	,! 9 ((E: 8)	i
$(\mathbf{n} + (\mathbf{m} + \mathbf{k})) = (\mathbf{n} + (\mathbf{k} + \mathbf{m}))$	,! 10 ( $\Rightarrow$ E: 7,9)	i

$(n + (k + m)) = j$	,! 11 (=E: 2,10)	i
$(n + (m + k)) = j \Rightarrow (n + (k + m)) = j$	,! 12 ( $\Rightarrow$ I: 2,11)	i
$( (n + (m + k)) = j \Rightarrow (n + (k + m)) = j )$	,! 13 (( )I: 12)	i
$\forall n \forall m \forall k \forall j ( (n + (m + k)) = j \Rightarrow (n + (k + m)) = j )$	! 14 ( $\forall$ I: 1,13)	i

□

! 13.

$\vdash \forall n \forall m \forall k \forall j ( j = (n + (m + k)) \Rightarrow j = (n + (k + m)) )$	i	
$n, m, k, j$	,! 1 (Prem)	i
$j = (n + (m + k))$	,! 2 (Prem)	i
$j = j$	,! 3 (=I)	i
$(n + (m + k)) = j$	,! 4 (=E: 2,3)	i
$( (n + (m + k)) = j \Rightarrow (n + (k + m)) = j )$	,! 5 ( $\forall$ E: P12)	i
$(n + (m + k)) = j \Rightarrow (n + (k + m)) = j$	,! 6 (( )E: 5)	i
$(n + (k + m)) = j$	,! 7 ( $\Rightarrow$ E: 4,6)	i
$j = (n + (k + m))$	,! 8 (=E: 3,7)	i
$j = (n + (m + k)) \Rightarrow j = (n + (k + m))$	,! 9 ( $\Rightarrow$ I: 2,8)	i
$( j = (n + (m + k)) \Rightarrow j = (n + (k + m)) )$	,! 10 (( )I: 9)	i
$\forall n \forall m \forall k \forall j ( j = (n + (m + k)) \Rightarrow j = (n + (k + m)) )$	! 11 ( $\forall$ I: 1,10)	i

□

! 14. The Associative Law of Addition

$\vdash \forall n \forall m \forall k ( \omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow (n + (m + k)) = ((n + m) + k) )$	i	
$n, m, k$	,! 1 (Prem)	i
$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$	,! 2 (Prem)	i
$\omega[n] \ \& \ \omega[m]$	,! 3 ( $\&$ E: 2)	i
$\omega[k]$	,! 4 ( $\&$ E: 2)	i

$( \omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}] \Rightarrow \omega[(\mathbf{n} + \mathbf{m})] )$  ,! 5 ( $\forall E$ : C1.8) ;  
 $\omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}] \Rightarrow \omega[(\mathbf{n} + \mathbf{m})]$  ,! 6 ( $(())E$ : 5) ;  
 $\omega[(\mathbf{n} + \mathbf{m})]$  ,! 7 ( $\Rightarrow E$ : 3,6) ;  
 $\omega[(\mathbf{n} + \mathbf{m})] \ \& \ \omega[\mathbf{k}]$  ,! 8 ( $\&I$ : 4,7) ;  
 $( \omega[(\mathbf{n} + \mathbf{m})] \ \& \ \omega[\mathbf{k}]$   
 $\Rightarrow \exists A \exists B ( \mathcal{N}[(\mathbf{n} + \mathbf{m}), A] \ \& \ \mathcal{N}[\mathbf{k}, B] \ \& \ (A \cap B) \equiv \phi$   
 $\ \& \ \mathcal{N}[(\mathbf{n} + \mathbf{m}) + \mathbf{k}], (A \cup B)] )$  )  
,! 9 ( $\forall E$ : C1.18;  
 $(\mathbf{n} + \mathbf{m}) + \mathbf{k}$ ): C1.7,8) ;  
 $\omega[(\mathbf{n} + \mathbf{m})] \ \& \ \omega[\mathbf{k}]$   
 $\Rightarrow \exists A \exists B ( \mathcal{N}[(\mathbf{n} + \mathbf{m}), A] \ \& \ \mathcal{N}[\mathbf{k}, B] \ \& \ (A \cap B) \equiv \phi$   
 $\ \& \ \mathcal{N}[(\mathbf{n} + \mathbf{m}) + \mathbf{k}], (A \cup B)] )$   
,! 10 ( $(())E$ : 9) ;  
 $\exists A \exists B ( \mathcal{N}[(\mathbf{n} + \mathbf{m}), A] \ \& \ \mathcal{N}[\mathbf{k}, B] \ \& \ (A \cap B) \equiv \phi$   
 $\ \& \ \mathcal{N}[(\mathbf{n} + \mathbf{m}) + \mathbf{k}], (A \cup B)] )$   
,! 11 ( $\Rightarrow E$ : 8,10) ;  
 $\exists B ( \mathcal{N}[(\mathbf{n} + \mathbf{m}), A] \ \& \ \mathcal{N}[\mathbf{k}, B] \ \& \ (A \cap B) \equiv \phi$   
 $\ \& \ \mathcal{N}[(\mathbf{n} + \mathbf{m}) + \mathbf{k}], (A \cup B)] )$   
,! 12 ( $\exists E$ : 11) ;  
 $( \mathcal{N}[(\mathbf{n} + \mathbf{m}), A] \ \& \ \mathcal{N}[\mathbf{k}, B] \ \& \ (A \cap B) \equiv \phi$   
 $\ \& \ \mathcal{N}[(\mathbf{n} + \mathbf{m}) + \mathbf{k}], (A \cup B)] )$   
,! 13 ( $\exists E$ : 12) ;  
 $\mathcal{N}[(\mathbf{n} + \mathbf{m}), A] \ \& \ \mathcal{N}[\mathbf{k}, B] \ \& \ (A \cap B) \equiv \phi$   
 $\ \& \ \mathcal{N}[(\mathbf{n} + \mathbf{m}) + \mathbf{k}], (A \cup B)]$   
,! 14 ( $(())E$ : 13) ;  
 $\mathcal{N}[(\mathbf{n} + \mathbf{m}), A]$  ,! 15 ( $\&E$ : 14) ;  
 $\mathcal{N}[\mathbf{k}, B]$  ,! 16 ( $\&E$ : 14) ;  
 $(A \cap B) \equiv \phi$  ,! 17 ( $\&E$ : 14) ;  
 $\mathcal{N}[(\mathbf{n} + \mathbf{m}) + \mathbf{k}], (A \cup B)]$  ,! 18 ( $\&E$ : 14) ;  
 $( \mathcal{N}[(\mathbf{n} + \mathbf{m}), A]$   
 $\Rightarrow \exists Q \exists R ( \mathcal{N}[\mathbf{n}, Q] \ \& \ \mathcal{N}[\mathbf{m}, R] \ \& \ (Q \cup R) \equiv \mathbf{A} \ \& \ (Q \cap R) \equiv \phi ) )$   
,! 19 ( $\forall E$ : C1.22) ;  
 $\mathcal{N}[(\mathbf{n} + \mathbf{m}), A]$   
 $\Rightarrow \exists Q \exists R ( \mathcal{N}[\mathbf{n}, Q] \ \& \ \mathcal{N}[\mathbf{m}, R] \ \& \ (Q \cup R) \equiv \mathbf{A} \ \& \ (Q \cap R) \equiv \phi )$   
,! 20 ( $(())E$ : 19) ;

$\exists Q \exists R ( \mathcal{N}[n, Q] \ \& \ \mathcal{N}[m, R] \ \& \ (Q \cup R) \equiv \mathbf{A} \ \& \ (Q \cap R) \equiv \phi )$   
, ! 21 ( $\Rightarrow$ E: 15, 20) i

$\exists R ( \mathcal{N}[n, Q] \ \& \ \mathcal{N}[m, R] \ \& \ (Q \cup R) \equiv \mathbf{A} \ \& \ (Q \cap R) \equiv \phi )$   
, ! 22 ( $\exists$ E: 21) i

$( \mathcal{N}[n, Q] \ \& \ \mathcal{N}[m, R] \ \& \ (Q \cup R) \equiv \mathbf{A} \ \& \ (Q \cap R) \equiv \phi )$   
, ! 23 ( $\exists$ E: 22) i

$\mathcal{N}[n, Q] \ \& \ \mathcal{N}[m, R] \ \& \ (Q \cup R) \equiv \mathbf{A} \ \& \ (Q \cap R) \equiv \phi$   
, ! 24 ( $(())$ E: 23) i

$(Q \cup R) \equiv \mathbf{A}$   
, ! 25 ( $\&$ E: 24) i

$R \subseteq (Q \cup R)$   
, ! 26 ( $\forall$ E: II2.13) i

$(Q \cup R) \equiv \mathbf{A} \ \& \ R \subseteq (Q \cup R)$   
, ! 27 ( $\&$ I: 25, 26) i

$( (Q \cup R) \equiv \mathbf{A} \ \& \ R \subseteq (Q \cup R) \Rightarrow R \subseteq \mathbf{A} )$   
, ! 28 ( $\forall$ E: II1.32) i

$(Q \cup R) \equiv \mathbf{A} \ \& \ R \subseteq (Q \cup R) \Rightarrow R \subseteq \mathbf{A}$   
, ! 29 ( $(())$ E: 28) i

$R \subseteq \mathbf{A}$   
, ! 30 ( $\Rightarrow$ E: 27, 29) i

$(\mathbf{A} \cap \mathbf{B}) \equiv \phi \ \& \ R \subseteq \mathbf{A}$   
, ! 31 ( $\&$ I: 17, 30) i

$( (\mathbf{A} \cap \mathbf{B}) \equiv \phi \ \& \ R \subseteq \mathbf{A} \Rightarrow (R \cap \mathbf{B}) \equiv \phi )$   
, ! 32 ( $\forall$ E: II5.37) i

$(\mathbf{A} \cap \mathbf{B}) \equiv \phi \ \& \ R \subseteq \mathbf{A} \Rightarrow (R \cap \mathbf{B}) \equiv \phi$   
, ! 33 ( $(())$ E: 32) i

$(R \cap \mathbf{B}) \equiv \phi$   
, ! 34 ( $\Rightarrow$ E: 31, 33) i

$\mathcal{N}[k, B] \ \& \ (R \cap \mathbf{B}) \equiv \phi$   
, ! 35 ( $\&$ I: 16, 34) i

$\mathcal{N}[m, R]$   
, ! 36 ( $\&$ E: 24) i

$\mathcal{N}[m, R] \ \& \ \mathcal{N}[k, B] \ \& \ (R \cap \mathbf{B}) \equiv \phi$   
, ! 37 ( $\&$ I: 35, 36) i

$\omega[m]$   
, ! 38 ( $\&$ E: 2) i

$\omega[m] \ \& \ \mathcal{N}[m, R] \ \& \ \mathcal{N}[k, B] \ \& \ (R \cap \mathbf{B}) \equiv \phi$   
, ! 39 ( $\&$ I: 37, 38) i

$\omega[m] \ \& \ \omega[k] \ \& \ \mathcal{N}[m, R] \ \& \ \mathcal{N}[k, B] \ \& \ (R \cap \mathbf{B}) \equiv \phi$   
, ! 40 ( $\&$ I: 4, 39) i

$( \omega[m] \ \& \ \omega[k] \ \& \ \mathcal{N}[m, R] \ \& \ \mathcal{N}[k, B] \ \& \ (R \cap \mathbf{B}) \equiv \phi$   
 $\Rightarrow \mathcal{N}[(m + k), (R \cup B)] )$   
, ! 41 ( $\forall$ E: C1.16) i

$\omega[m] \ \& \ \omega[k] \ \& \ \mathcal{N}[m, R] \ \& \ \mathcal{N}[k, B] \ \& \ (R \cap \mathbf{B}) \equiv \phi$

$\Rightarrow \mathcal{N}[(\mathbf{m} + \mathbf{k}), (\mathbf{R} \cup \mathbf{B})]$	,! 42 (( )E: 41)	i
$\mathcal{N}[(\mathbf{m} + \mathbf{k}), (\mathbf{R} \cup \mathbf{B})]$	,! 43 ( $\Rightarrow$ E: 40,42)	i
$(\mathbf{Q} \cap \mathbf{R}) \equiv \phi$	,! 44 (&E: 24)	i
$\mathbf{Q} \subseteq (\mathbf{Q} \cup \mathbf{R})$	,! 45 ( $\forall$ E: II2.12)	i
$(\mathbf{Q} \cup \mathbf{R}) \equiv \mathbf{A} \ \& \ \mathbf{Q} \subseteq (\mathbf{Q} \cup \mathbf{R})$	,! 46 (&I: 25,45)	i
$( (\mathbf{Q} \cup \mathbf{R}) \equiv \mathbf{A} \ \& \ \mathbf{Q} \subseteq (\mathbf{Q} \cup \mathbf{R}) \Rightarrow \mathbf{Q} \subseteq \mathbf{A} )$	,! 47 ( $\forall$ E: III1.32)	i
$(\mathbf{Q} \cup \mathbf{R}) \equiv \mathbf{A} \ \& \ \mathbf{Q} \subseteq (\mathbf{Q} \cup \mathbf{R}) \Rightarrow \mathbf{Q} \subseteq \mathbf{A}$	,! 48 (( )E: 47)	i
$\mathbf{Q} \subseteq \mathbf{A}$	,! 49 ( $\Rightarrow$ E: 46,48)	i
$(\mathbf{A} \cap \mathbf{B}) \equiv \phi \ \& \ \mathbf{Q} \subseteq \mathbf{A}$	,! 50 (&I: 17,49)	i
$( (\mathbf{A} \cap \mathbf{B}) \equiv \phi \ \& \ \mathbf{Q} \subseteq \mathbf{A} \Rightarrow (\mathbf{Q} \cap \mathbf{B}) \equiv \phi )$	,! 51 ( $\forall$ E: II5.37)	i
$(\mathbf{A} \cap \mathbf{B}) \equiv \phi \ \& \ \mathbf{Q} \subseteq \mathbf{A} \Rightarrow (\mathbf{Q} \cap \mathbf{B}) \equiv \phi$	,! 52 (( )E: 51)	i
$(\mathbf{Q} \cap \mathbf{B}) \equiv \phi$	,! 53 ( $\Rightarrow$ E: 50,52)	i
$(\mathbf{Q} \cap \mathbf{R}) \equiv \phi \ \& \ (\mathbf{Q} \cap \mathbf{B}) \equiv \phi$	,! 54 (&I: 44,53)	i
$( (\mathbf{Q} \cap \mathbf{R}) \equiv \phi \ \& \ (\mathbf{Q} \cap \mathbf{B}) \equiv \phi \Rightarrow (\mathbf{Q} \cap (\mathbf{R} \cup \mathbf{B})) \equiv \phi )$	,! 55 ( $\forall$ E: II5.39)	i
$(\mathbf{Q} \cap \mathbf{R}) \equiv \phi \ \& \ (\mathbf{Q} \cap \mathbf{B}) \equiv \phi \Rightarrow (\mathbf{Q} \cap (\mathbf{R} \cup \mathbf{B})) \equiv \phi$	,! 56 (( )E: 55)	i
$(\mathbf{Q} \cap (\mathbf{R} \cup \mathbf{B})) \equiv \phi$	,! 57 ( $\Rightarrow$ E: 54,56)	i
$\mathcal{N}[(\mathbf{m} + \mathbf{k}), (\mathbf{R} \cup \mathbf{B})] \ \& \ (\mathbf{Q} \cap (\mathbf{R} \cup \mathbf{B})) \equiv \phi$	,! 58 (&I: 43,57)	i
$\mathcal{N}[\mathbf{n}, \mathbf{Q}]$	,! 59 (&E: 24)	i
$\mathcal{N}[\mathbf{n}, \mathbf{Q}] \ \& \ \mathcal{N}[(\mathbf{m} + \mathbf{k}), (\mathbf{R} \cup \mathbf{B})] \ \& \ (\mathbf{Q} \cap (\mathbf{R} \cup \mathbf{B})) \equiv \phi$	,! 60 (&I: 58,59)	i
$\omega[\mathbf{n}]$	,! 61 (&E: 2)	i
$\omega[\mathbf{m}] \ \& \ \omega[\mathbf{k}]$	,! 62 (&I: 4,38)	i
$( \omega[\mathbf{m}] \ \& \ \omega[\mathbf{k}] \Rightarrow \omega[(\mathbf{m} + \mathbf{k})] )$	,! 63 ( $\forall$ E: C1.8)	i
$\omega[\mathbf{m}] \ \& \ \omega[\mathbf{k}] \Rightarrow \omega[(\mathbf{m} + \mathbf{k})]$	,! 64 (( )E: 63)	i
$\omega[(\mathbf{m} + \mathbf{k})]$	,! 65 ( $\Rightarrow$ E: 62,64)	i

$\omega[\mathbf{n}] \ \& \ \omega[(\mathbf{m} + \mathbf{k})]$  ,! 66 (&I: 61,65) ;

$\omega[\mathbf{n}] \ \& \ \omega[(\mathbf{m} + \mathbf{k})] \ \& \ \mathfrak{N}[\mathbf{n},\mathbf{Q}] \ \& \ \mathfrak{N}[(\mathbf{m} + \mathbf{k}),(\mathbf{R} \cup \mathbf{B})]$   
&  $(\mathbf{Q} \cap (\mathbf{R} \cup \mathbf{B})) \equiv \phi$  ,! 67 (&I: 60,66) ;

$(\ \omega[\mathbf{n}] \ \& \ \omega[(\mathbf{m} + \mathbf{k})] \ \& \ \mathfrak{N}[\mathbf{n},\mathbf{Q}] \ \& \ \mathfrak{N}[(\mathbf{m} + \mathbf{k}),(\mathbf{R} \cup \mathbf{B})]$   
&  $(\mathbf{Q} \cap (\mathbf{R} \cup \mathbf{B})) \equiv \phi$   
 $\Rightarrow \mathfrak{N}[(\mathbf{n} + (\mathbf{m} + \mathbf{k})),(\mathbf{Q} \cup (\mathbf{R} \cup \mathbf{B}))]$  )  
,! 68 ( $\forall E$ : C1.16;  
 $(\mathbf{m} + \mathbf{k})$ : C1.7,66) ;

$\omega[\mathbf{n}] \ \& \ \omega[(\mathbf{m} + \mathbf{k})] \ \& \ \mathfrak{N}[\mathbf{n},\mathbf{Q}] \ \& \ \mathfrak{N}[(\mathbf{m} + \mathbf{k}),(\mathbf{R} \cup \mathbf{B})]$   
&  $(\mathbf{Q} \cap (\mathbf{R} \cup \mathbf{B})) \equiv \phi$   
 $\Rightarrow \mathfrak{N}[(\mathbf{n} + (\mathbf{m} + \mathbf{k})),(\mathbf{Q} \cup (\mathbf{R} \cup \mathbf{B}))]$  )  
,! 69 (( $E$ : 68) ;

$\mathfrak{N}[(\mathbf{n} + (\mathbf{m} + \mathbf{k})),(\mathbf{Q} \cup (\mathbf{R} \cup \mathbf{B}))]$  ,! 70 ( $\Rightarrow E$ : 67,69) ;

$(\ \omega[\mathbf{n}] \ \& \ \omega[(\mathbf{m} + \mathbf{k})] \Rightarrow \omega[(\mathbf{n} + (\mathbf{m} + \mathbf{k}))]$  )  
,! 71 ( $\forall E$ : C1.8) ;

$\omega[\mathbf{n}] \ \& \ \omega[(\mathbf{m} + \mathbf{k})] \Rightarrow \omega[(\mathbf{n} + (\mathbf{m} + \mathbf{k}))]$  ,! 72 (( $E$ : 71) ;

$\omega[(\mathbf{n} + (\mathbf{m} + \mathbf{k}))]$  ,! 73 ( $\Rightarrow E$ : 66,72) ;

$\omega[(\mathbf{n} + (\mathbf{m} + \mathbf{k}))] \ \& \ \mathfrak{N}[(\mathbf{n} + (\mathbf{m} + \mathbf{k})),(\mathbf{Q} \cup (\mathbf{R} \cup \mathbf{B}))]$   
,! 74 (&I: 70,73) ;

$((\mathbf{Q} \cup \mathbf{R}) \cup \mathbf{B}) \equiv (\mathbf{Q} \cup (\mathbf{R} \cup \mathbf{B}))$  ,! 75 ( $\forall E$ : II2.54) ;

$(\ (\mathbf{Q} \cup \mathbf{R}) \equiv \mathbf{A} \Rightarrow ((\mathbf{Q} \cup \mathbf{R}) \cup \mathbf{B}) \equiv (\mathbf{A} \cup \mathbf{B}))$  )  
,! 76 ( $\forall E$ : II2.36) ;

$(\mathbf{Q} \cup \mathbf{R}) \equiv \mathbf{A} \Rightarrow ((\mathbf{Q} \cup \mathbf{R}) \cup \mathbf{B}) \equiv (\mathbf{A} \cup \mathbf{B})$   
,! 77 (( $E$ : 76) ;

$((\mathbf{Q} \cup \mathbf{R}) \cup \mathbf{B}) \equiv (\mathbf{A} \cup \mathbf{B})$  ,! 78 ( $\Rightarrow E$ : 25,77) ;

$((\mathbf{Q} \cup \mathbf{R}) \cup \mathbf{B}) \equiv (\mathbf{Q} \cup (\mathbf{R} \cup \mathbf{B}))$   
&  $((\mathbf{Q} \cup \mathbf{R}) \cup \mathbf{B}) \equiv (\mathbf{A} \cup \mathbf{B})$  ,! 79 (&I: 75,78) ;

$(\ ((\mathbf{Q} \cup \mathbf{R}) \cup \mathbf{B}) \equiv (\mathbf{Q} \cup (\mathbf{R} \cup \mathbf{B}))$   
&  $((\mathbf{Q} \cup \mathbf{R}) \cup \mathbf{B}) \equiv (\mathbf{A} \cup \mathbf{B})$   
 $\Rightarrow (\mathbf{Q} \cup (\mathbf{R} \cup \mathbf{B})) \equiv (\mathbf{A} \cup \mathbf{B}))$  )  
,! 80 ( $\forall E$ : II1.19) ;

$((\mathbf{Q} \cup \mathbf{R}) \cup \mathbf{B}) \equiv (\mathbf{Q} \cup (\mathbf{R} \cup \mathbf{B}))$   
&  $((\mathbf{Q} \cup \mathbf{R}) \cup \mathbf{B}) \equiv (\mathbf{A} \cup \mathbf{B})$   
 $\Rightarrow (\mathbf{Q} \cup (\mathbf{R} \cup \mathbf{B})) \equiv (\mathbf{A} \cup \mathbf{B})$  )  
,! 81 (( $E$ : 80) ;

$(Q \cup (R \cup B)) \equiv (A \cup B)$  ,! 82 ( $\Rightarrow$ E: 79,81) ;

$\omega[(n + (m + k))] \& \mathfrak{N}[(n + (m + k)), (Q \cup (R \cup B))]$   
 $\& (Q \cup (R \cup B)) \equiv (A \cup B)$  ,! 83 (&I: 74,82) ;

$(\omega[(n + (m + k))] \& \mathfrak{N}[(n + (m + k)), (Q \cup (R \cup B))])$   
 $\& (Q \cup (R \cup B)) \equiv (A \cup B)$   
 $\Rightarrow \mathfrak{N}[(n + (m + k)), (A \cup B)]$  )  
 ,! 84 ( $\forall$ E: IV4.5;  
( $n + (m + k)$ ): C1.7,66)  
 ;

$\omega[(n + (m + k))] \& \mathfrak{N}[(n + (m + k)), (Q \cup (R \cup B))]$   
 $\& (Q \cup (R \cup B)) \equiv (A \cup B)$   
 $\Rightarrow \mathfrak{N}[(n + (m + k)), (A \cup B)]$  )  
 ,! 85 (( )E: 84) ;

$\mathfrak{N}[(n + (m + k)), (A \cup B)]$  ,! 86 ( $\Rightarrow$ E: 83,85) ;

$\mathfrak{N}[(n + (m + k)), (A \cup B)] \& \mathfrak{N}[((n + m) + k), (A \cup B)]$   
 ,! 87 (&I: 18,86) ;

$(\omega[(n + m)] \& \omega[k] \Rightarrow \omega[((n + m) + k])$  )  
 ,! 88 ( $\forall$ E: C1.8) ;

$\omega[(n + m)] \& \omega[k] \Rightarrow \omega[((n + m) + k)]$  ,! 89 (( )E: 88) ;

$\omega[((n + m) + k)]$  ,! 90 ( $\Rightarrow$ E: 8,89) ;

$\omega[(n + (m + k))] \& \omega[((n + m) + k)]$  ,! 91 (&I: 73,90) ;

$\omega[(n + (m + k))] \& \omega[((n + m) + k)]$   
 $\& \mathfrak{N}[(n + (m + k)), (A \cup B)] \& \mathfrak{N}[((n + m) + k), (A \cup B)]$   
 ,! 92 (&I: 87,91) ;

$(\omega[(n + (m + k))] \& \omega[((n + m) + k)])$   
 $\& \mathfrak{N}[(n + (m + k)), (A \cup B)] \& \mathfrak{N}[((n + m) + k), (A \cup B)]$   
 $\Rightarrow (n + (m + k)) = ((n + m) + k)$  )  
 ,! 93 ( $\forall$ E: IV2.10;  
( $n + (m + k)$ ): C1.7,66;  
( $(n + m) + k$ ): C1.7,8)  
 ;

$\omega[(n + (m + k))] \& \omega[((n + m) + k)]$   
 $\& \mathfrak{N}[(n + (m + k)), (A \cup B)] \& \mathfrak{N}[((n + m) + k), (A \cup B)]$   
 $\Rightarrow (n + (m + k)) = ((n + m) + k)$   
 ,! 94 (( )E: 93) ;

$(n + (m + k)) = ((n + m) + k)$  ,! 95 ( $\Rightarrow$ E: 92,94) ;

$\omega[n] \& \omega[m] \& \omega[k] \Rightarrow (n + (m + k)) = ((n + m) + k)$

,! 96 ( $\Rightarrow$ I: 2,95) i

(  $\omega[n]$  &  $\omega[m]$  &  $\omega[k]$   $\Rightarrow$  (  $n + (m + k)$  ) = ( (  $n + m$  ) +  $k$  ) )  
,! 97 (())I: 96) i

$\forall n \forall m \forall k$  (  $\omega[n]$  &  $\omega[m]$  &  $\omega[k]$   $\Rightarrow$  (  $n + (m + k)$  ) = ( (  $n + m$  ) +  $k$  ) )  
! 98 ( $\forall$ I: 1,97) i

□

! P15 through P18 are applications of the Associative Law. i

! 15. i

$\vdash \forall n \forall m \forall k \forall j$  ( (  $n + (m + k)$  ) =  $j$   $\Rightarrow$  ( (  $n + m$  ) +  $k$  ) =  $j$  ) i

**$n, m, k, j$**  ,! 1 (Prem) i

(  **$n + (m + k)$**  ) =  **$j$**  ,! 2 (Prem) i

$\omega[n]$  &  $\omega[(m + k)]$  ,! 3 ( $\mathbb{T}$ E: C1.7,2) i

$\omega[(m + k)]$  ,! 4 (&E: 3) i

$\omega[m]$  &  $\omega[k]$  ,! 5 ( $\mathbb{T}$ E: C1.7,4) i

$\omega[n]$  ,! 6 (&E: 3) i

$\omega[n]$  &  $\omega[m]$  &  $\omega[k]$  ,! 7 (&I: 5,6) i

(  $\omega[n]$  &  $\omega[m]$  &  $\omega[k]$   $\Rightarrow$  (  $n + (m + k)$  ) = ( (  $n + m$  ) +  $k$  ) )  
,! 8 ( $\forall$ E: P14) i

$\omega[n]$  &  $\omega[m]$  &  $\omega[k]$   $\Rightarrow$  (  $n + (m + k)$  ) = ( (  $n + m$  ) +  $k$  )  
,! 9 (())E: 8) i

(  $n + (m + k)$  ) = ( (  $n + m$  ) +  $k$  )  
,! 10 ( $\Rightarrow$ E: 7,9) i

( (  $n + m$  ) +  $k$  ) =  $j$  ,! 11 (=E: 2,10) i

(  $n + (m + k)$  ) =  $j$   $\Rightarrow$  ( (  $n + m$  ) +  $k$  ) =  $j$  )  
,! 12 ( $\Rightarrow$ I: 2,11) i

( (  $n + (m + k)$  ) =  $j$   $\Rightarrow$  ( (  $n + m$  ) +  $k$  ) =  $j$  )  
,! 13 (())I: 12) i

$\forall n \forall m \forall k \forall j$  ( (  $n + (m + k)$  ) =  $j$   $\Rightarrow$  ( (  $n + m$  ) +  $k$  ) =  $j$  )  
! 14 ( $\forall$ I: 1,13) i

□

! 16. i

$\vdash \forall n \forall m \forall k \forall j$  ( ( (  $n + m$  ) +  $k$  ) =  $j$   $\Rightarrow$  (  $n + (m + k)$  ) =  $j$  ) i

$n, m, k, j$	,! 1 (Prem)	i
$((n + m) + k) = j$	,! 2 (Prem)	i
$\omega[(n + m)] \ \& \ \omega[k]$	,! 3 (TE: C1.7,2)	i
$\omega[(n + m)]$	,! 4 (&E: 3)	i
$\omega[n] \ \& \ \omega[m]$	,! 5 (TE: C1.7,4)	i
$\omega[k]$	,! 6 (&E: 3)	i
$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$	,! 7 (&I: 5,6)	i
$(\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow (n + (m + k))) = ((n + m) + k)$	,! 8 ( $\forall$ E: P14)	i
$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow (n + (m + k)) = ((n + m) + k)$	,! 9 (()E: 8)	i
$(n + (m + k)) = ((n + m) + k)$	,! 10 ( $\Rightarrow$ E: 7,9)	i
$(n + (m + k)) = j$	,! 11 (=E: 2,10)	i
$((n + m) + k) = j \Rightarrow (n + (m + k)) = j$	,! 12 ( $\Rightarrow$ I: 2,11)	i
$(((n + m) + k) = j \Rightarrow (n + (m + k)) = j)$	,! 13 (()I: 12)	i
$\forall n \forall m \forall k \forall j ((n + m) + k) = j \Rightarrow (n + (m + k)) = j$	! 14 ( $\forall$ I: 1,13)	i

□

! 17.

$\vdash \forall n \forall m \forall k \forall j (j = (n + (m + k)) \Rightarrow j = ((n + m) + k))$		i
$n, m, k, j$	,! 1 (Prem)	i
$j = (n + (m + k))$	,! 2 (Prem)	i
$j = j$	,! 3 (=I)	i
$(n + (m + k)) = j$	,! 4 (=E: 2,3)	i
$((n + (m + k)) = j \Rightarrow ((n + m) + k) = j)$	,! 5 ( $\forall$ E: P15)	i
$(n + (m + k)) = j \Rightarrow ((n + m) + k) = j$	,! 6 (()E: 5)	i
$((n + m) + k) = j$	,! 7 ( $\Rightarrow$ E: 4,6)	i
$j = ((n + m) + k)$	,! 8 (=E: 3,7)	i
$j = (n + (m + k)) \Rightarrow j = ((n + m) + k)$	,! 9 ( $\Rightarrow$ I: 2,8)	i

$( j = ( n + ( m + k ) ) \Rightarrow j = ( ( n + m ) + k ) )$   
 ,! 10 ((I: 9) i

$\forall n \forall m \forall k \forall j ( j = ( n + ( m + k ) ) \Rightarrow j = ( ( n + m ) + k ) )$   
 ! 11 ( $\forall$ I: 1,10) i

□

! 18. i

$\vdash \forall n \forall m \forall k \forall j ( j = ( ( n + m ) + k ) \Rightarrow j = ( n + ( m + k ) ) )$  i

$n, m, k, j$  ,! 1 (Prem) i

$j = ( ( n + m ) + k )$  ,! 2 (Prem) i

$j = j$  ,! 3 (=I) i

$( ( n + m ) + k ) = j$  ,! 4 (=E: 2,3) i

$( ( ( n + m ) + k ) = j \Rightarrow ( n + ( m + k ) ) = j )$   
 ,! 5 ( $\forall$ E: P16) i

$( ( n + m ) + k ) = j \Rightarrow ( n + ( m + k ) ) = j$   
 ,! 6 ((E: 5) i

$( n + ( m + k ) ) = j$  ,! 7 ( $\Rightarrow$ E: 4,6) i

$j = ( n + ( m + k ) )$  ,! 8 (=E: 3,7) i

$j = ( ( n + m ) + k ) \Rightarrow j = ( n + ( m + k ) )$  ,! 9 ( $\Rightarrow$ I: 2,8) i

$( j = ( ( n + m ) + k ) \Rightarrow j = ( n + ( m + k ) ) )$   
 ,! 10 ((I: 9) i

$\forall n \forall m \forall k \forall j ( j = ( ( n + m ) + k ) \Rightarrow j = ( n + ( m + k ) ) )$   
 ! 11 ( $\forall$ I: 1,10) i

□

! P19 through P29 are applications of the Commutative and Associative Laws. i

! 19. i

$\vdash \forall n \forall m \forall k ( \omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$   
 $\Rightarrow ( ( n + m ) + k ) = ( ( n + k ) + m ) )$  i

$n, m, k$  ,! 1 (Prem) i

$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$  ,! 2 (Prem) i

$( \omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow ( n + ( m + k ) ) = ( ( n + m ) + k ) )$   
 ,! 3 ( $\forall$ E: P14) i

$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow ( n + ( m + k ) ) = ( ( n + m ) + k )$

	,! 4 ( )E: 3	i
$(n + (m + k)) = ((n + m) + k)$	,! 5 ( $\Rightarrow$ E: 2,4)	i
$((n + m) + k) = ((n + m) + k)$	,! 6 (=E: 5,5)	i
$((n + m) + k) = (n + (m + k))$	,! 7 (=E: 5,6)	i
$\omega[(n + m)] \ \& \ \omega[k]$	,! 8 ( $\mathbb{T}$ E: C1.7,7)	i
$((n + m) + k) = (n + (m + k))$ $\Rightarrow ((n + m) + k) = (n + (k + m))$	,! 9 ( $\forall$ E: P13; $((n + m) + k)$ : C1.7,8)	i
$((n + m) + k) = (n + (m + k))$ $\Rightarrow ((n + m) + k) = (n + (k + m))$	,! 10 ( )E: 9	i
$((n + m) + k) = (n + (k + m))$	,! 11 ( $\Rightarrow$ E: 7,10)	i
$((n + m) + k) = (n + (k + m))$ $\Rightarrow ((n + m) + k) = ((n + k) + m)$	,! 12 ( $\forall$ E: P17; $((n + m) + k)$ : C1.7,8)	i
$((n + m) + k) = (n + (k + m))$ $\Rightarrow ((n + m) + k) = ((n + k) + m)$	,! 13 ( )E: 12	i
$((n + m) + k) = ((n + k) + m)$	,! 14 ( $\Rightarrow$ E: 11,13)	i
$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow ((n + m) + k) = ((n + k) + m)$	,! 15 ( $\Rightarrow$ I: 2,14)	i
$(\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow ((n + m) + k) = ((n + k) + m))$	,! 16 ( )I: 15	i
$\forall n \forall m \forall k (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$ $\Rightarrow ((n + m) + k) = ((n + k) + m))$	! 17 ( $\forall$ I: 1,16)	i

□

! 20.

$\vdash \forall n \forall m \forall k \forall j ((n + m) + k) = j \Rightarrow ((n + k) + m) = j$		i
<b>n, m, k, j</b>	,! 1 (Prem)	i
$((n + m) + k) = j$	,! 2 (Prem)	i
$\omega[(n + m)] \ \& \ \omega[k]$	,! 3 ( $\mathbb{T}$ E: C1.7,2)	i
$\omega[(n + m)]$	,! 4 ( $\&$ E: 3)	i

$\omega[\mathbf{k}]$	, ! 5 (&E: 3)	i
$\omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}]$	, ! 6 ( $\mathbb{T}E$ : C1.7,4)	i
$\omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}] \ \& \ \omega[\mathbf{k}]$	, ! 7 (&I: 5,6)	i
$( \ \omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}] \ \& \ \omega[\mathbf{k}] \Rightarrow ((\mathbf{n} + \mathbf{m}) + \mathbf{k}) = ((\mathbf{n} + \mathbf{k}) + \mathbf{m}) )$	, ! 8 ( $\forall E$ : P19)	i
$\omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}] \ \& \ \omega[\mathbf{k}] \Rightarrow ((\mathbf{n} + \mathbf{m}) + \mathbf{k}) = ((\mathbf{n} + \mathbf{k}) + \mathbf{m})$	, ! 9 (()E: 8)	i
$((\mathbf{n} + \mathbf{m}) + \mathbf{k}) = ((\mathbf{n} + \mathbf{k}) + \mathbf{m})$	, ! 10 ( $\Rightarrow E$ : 7,9)	i
$((\mathbf{n} + \mathbf{k}) + \mathbf{m}) = \mathbf{j}$	, ! 11 (=E: 2,10)	i
$((\mathbf{n} + \mathbf{m}) + \mathbf{k}) = \mathbf{j} \Rightarrow ((\mathbf{n} + \mathbf{k}) + \mathbf{m}) = \mathbf{j}$	, ! 12 ( $\Rightarrow I$ : 2,11)	i
$( ((\mathbf{n} + \mathbf{m}) + \mathbf{k}) = \mathbf{j} \Rightarrow ((\mathbf{n} + \mathbf{k}) + \mathbf{m}) = \mathbf{j} )$	, ! 13 (()I: 12)	i
$\forall n \forall m \forall k \forall j ( ((\mathbf{n} + \mathbf{m}) + \mathbf{k}) = \mathbf{j} \Rightarrow ((\mathbf{n} + \mathbf{k}) + \mathbf{m}) = \mathbf{j} )$	! 14 ( $\forall I$ : 1,13)	i
$\square$		
! 21.		
$\vdash \forall n \forall m \forall k \forall j ( j = ((\mathbf{n} + \mathbf{m}) + \mathbf{k}) \Rightarrow j = ((\mathbf{n} + \mathbf{k}) + \mathbf{m}) )$		i
$\mathbf{n}, \mathbf{m}, \mathbf{k}, \mathbf{j}$	, ! 1 (Prem)	i
$\mathbf{j} = ((\mathbf{n} + \mathbf{m}) + \mathbf{k})$	, ! 2 (Prem)	i
$\mathbf{j} = \mathbf{j}$	, ! 3 (=I)	i
$((\mathbf{n} + \mathbf{m}) + \mathbf{k}) = \mathbf{j}$	, ! 4 (=E: 2,3)	i
$( ((\mathbf{n} + \mathbf{m}) + \mathbf{k}) = \mathbf{j} \Rightarrow ((\mathbf{n} + \mathbf{k}) + \mathbf{m}) = \mathbf{j} )$	, ! 5 ( $\forall E$ : P20)	i
$((\mathbf{n} + \mathbf{m}) + \mathbf{k}) = \mathbf{j} \Rightarrow ((\mathbf{n} + \mathbf{k}) + \mathbf{m}) = \mathbf{j}$	, ! 6 (()E: 5)	i
$((\mathbf{n} + \mathbf{k}) + \mathbf{m}) = \mathbf{j}$	, ! 7 ( $\Rightarrow E$ : 4,6)	i
$\mathbf{j} = ((\mathbf{n} + \mathbf{k}) + \mathbf{m})$	, ! 8 (=E: 3,7)	i
$\mathbf{j} = ((\mathbf{n} + \mathbf{m}) + \mathbf{k}) \Rightarrow \mathbf{j} = ((\mathbf{n} + \mathbf{k}) + \mathbf{m})$	, ! 9 ( $\Rightarrow I$ : 2,8)	i
$( \mathbf{j} = ((\mathbf{n} + \mathbf{m}) + \mathbf{k}) \Rightarrow \mathbf{j} = ((\mathbf{n} + \mathbf{k}) + \mathbf{m}) )$	, ! 10 (()I: 9)	i
$\forall n \forall m \forall k \forall j ( j = ((\mathbf{n} + \mathbf{m}) + \mathbf{k}) \Rightarrow j = ((\mathbf{n} + \mathbf{k}) + \mathbf{m}) )$	! 11 ( $\forall I$ : 1,10)	i

□

! 22.

		i
$\vdash \forall n \forall m \forall k ( \omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$		
$\Rightarrow (n + (m + k)) = (m + (n + k)) )$		i
<b>n,m,k</b>	,! 1 (Prem)	i
$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$	,! 2 (Prem)	i
$( \omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow (n + (m + k)) = ((n + m) + k) )$	,! 3 ( $\forall E$ : P14)	i
$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow (n + (m + k)) = ((n + m) + k)$	,! 4 ( $()E$ : 3)	i
$(n + (m + k)) = ((n + m) + k)$	,! 5 ( $\Rightarrow E$ : 2,4)	i
$\omega[n] \ \& \ \omega[(m + k)]$	,! 6 ( $\mathbb{T}E$ : C1.7,5)	i
$( (n + (m + k)) = ((n + m) + k)$		
$\Rightarrow (n + (m + k)) = ((m + n) + k) )$	,! 7 ( $\forall E$ : P10; $(n + (m + k))$ : C1.7,6)	i
$(n + (m + k)) = ((n + m) + k)$		
$\Rightarrow (n + (m + k)) = ((m + n) + k)$	,! 8 ( $()E$ : 7)	i
$(n + (m + k)) = ((m + n) + k)$	,! 9 ( $\Rightarrow E$ : 5,8)	i
$( (n + (m + k)) = ((m + n) + k)$		
$\Rightarrow (n + (m + k)) = (m + (n + k)) )$	,! 10 ( $\forall E$ : P18; $(n + (m + k))$ : C1.7,6)	i
$(n + (m + k)) = ((m + n) + k)$		
$\Rightarrow (n + (m + k)) = (m + (n + k))$	,! 11 ( $()E$ : 10)	i
$(n + (m + k)) = (m + (n + k))$	,! 12 ( $\Rightarrow E$ : 9,11)	i
$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow (n + (m + k)) = (m + (n + k))$	,! 13 ( $\Rightarrow I$ : 2,12)	i
$( \omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow (n + (m + k)) = (m + (n + k)) )$	,! 14 ( $()I$ : 13)	i
$\forall n \forall m \forall k ( \omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$		
$\Rightarrow (n + (m + k)) = (m + (n + k)) )$	! 15 ( $\forall I$ : 1,14)	i

□

! 23.

$\vdash \forall n \forall m \forall k \forall j ( (n + (m + k)) = j \Rightarrow (m + (n + k)) = j )$		i
<b>n, m, k, j</b>	,! 1 (Prem)	i
$(n + (m + k)) = j$	,! 2 (Prem)	i
$\omega[n] \ \& \ \omega[(m + k)]$	,! 3 ( $\mathbb{T}\mathbb{E}$ : C1.7,2)	i
$\omega[n]$	,! 4 ( $\&\mathbb{E}$ : 3)	i
$\omega[(m + k)]$	,! 5 ( $\&\mathbb{E}$ : 3)	i
$\omega[m] \ \& \ \omega[k]$	,! 6 ( $\mathbb{T}\mathbb{E}$ : C1.7,5)	i
$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$	,! 7 ( $\&\mathbb{I}$ : 4,6)	i
$( \omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow (n + (m + k)) = (m + (n + k)) )$	,! 8 ( $\forall\mathbb{E}$ : P22)	i
$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow (n + (m + k)) = (m + (n + k))$	,! 9 ( $()\mathbb{E}$ : 8)	i
$(n + (m + k)) = (m + (n + k))$	,! 10 ( $\Rightarrow\mathbb{E}$ : 7,9)	i
$(m + (n + k)) = j$	,! 11 ( $=\mathbb{E}$ : 2,10)	i
$(n + (m + k)) = j \Rightarrow (m + (n + k)) = j$	,! 12 ( $\Rightarrow\mathbb{I}$ : 2,11)	i
$( (n + (m + k)) = j \Rightarrow (m + (n + k)) = j )$	,! 13 ( $()\mathbb{I}$ : 12)	i
$\forall n \forall m \forall k \forall j ( (n + (m + k)) = j \Rightarrow (m + (n + k)) = j )$	! 14 ( $\forall\mathbb{I}$ : 1,13)	i

□

! 24.

$\vdash \forall n \forall m \forall k \forall j ( j = (n + (m + k)) \Rightarrow j = (m + (n + k)) )$		i
<b>n, m, k, j</b>	,! 1 (Prem)	i
$j = (n + (m + k))$	,! 2 (Prem)	i
$j = j$	,! 3 ( $=\mathbb{I}$ )	i
$(n + (m + k)) = j$	,! 4 ( $=\mathbb{E}$ : 2,3)	i
$( (n + (m + k)) = j \Rightarrow (m + (n + k)) = j )$	,! 5 ( $\forall\mathbb{E}$ : P23)	i
$(n + (m + k)) = j \Rightarrow (m + (n + k)) = j$	,! 6 ( $()\mathbb{E}$ : 5)	i

$$\begin{array}{l}
(m + (n + k)) = j \quad ,! 7 (\Rightarrow E: 4,6) \quad i \\
j = (m + (n + k)) \quad ,! 8 (=E: 3,7) \quad i \\
j = (n + (m + k)) \Rightarrow j = (m + (n + k)) \quad ,! 9 (\Rightarrow I: 2,8) \quad i \\
( j = (n + (m + k)) \Rightarrow j = (m + (n + k)) ) \\
\quad ,! 10 ((I: 9) \quad i \\
\forall n \forall m \forall k \forall j ( j = (n + (m + k)) \Rightarrow j = (m + (n + k)) ) \\
\quad ! 11 (\forall I: 1,10) \quad i
\end{array}$$

□

! 25.

$$\begin{array}{l}
\vdash \forall n \forall m \forall k ( \omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \\
\quad \Rightarrow ((n + m) + k) = ((k + n) + m) ) \quad i \\
n, m, k \quad ,! 1 (Prem) \quad i \\
\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \quad ,! 2 (Prem) \quad i \\
( \omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \\
\quad \Rightarrow ((n + m) + k) = ((n + k) + m) ) \\
\quad ,! 3 (\forall E: P19) \quad i \\
\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \\
\Rightarrow ((n + m) + k) = ((n + k) + m) \\
\quad ,! 4 ((E: 3) \quad i \\
((n + m) + k) = ((n + k) + m) \quad ,! 5 (\Rightarrow E: 2,4) \quad i \\
\omega[(n + m)] \ \& \ \omega[k] \quad ,! 6 (\mathbb{T}E: C1.7,5) \quad i \\
( ((n + m) + k) = ((n + k) + m) \\
\quad \Rightarrow ((n + m) + k) = ((k + n) + m) ) \\
\quad ,! 7 (\forall E: P10; \\
\quad ((n + m) + k): C1.7,6) \quad i \\
((n + m) + k) = ((n + k) + m) \\
\Rightarrow ((n + m) + k) = ((k + n) + m) \\
\quad ,! 8 ((E: 7) \quad i \\
((n + m) + k) = ((k + n) + m) \quad ,! 9 (\Rightarrow E: 5,8) \quad i \\
\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow ((n + m) + k) = ((k + n) + m) \\
\quad ,! 10 (\Rightarrow I: 2,9) \quad i \\
( \omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow ((n + m) + k) = ((k + n) + m) ) \\
\quad ,! 11 ((I: 10) \quad i \\
\forall n \forall m \forall k ( \omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow ((n + m) + k) = ((k + n) + m) )
\end{array}$$

! 12 ( $\forall I: 1,11$ ) i

□

! 26. i

$\vdash \forall n \forall m \forall k ( \omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$   
 $\Rightarrow (n + (m + k)) = (m + (k + n)) )$  i

**n,m,k** ,! 1 (Prem) i

$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$  ,! 2 (Prem) i

$( \omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$   
 $\Rightarrow (n + (m + k)) = (m + (n + k)) )$   
 ,! 3 ( $\forall E: P22$ ) i

$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$   
 $\Rightarrow (n + (m + k)) = (m + (n + k))$   
 ,! 4 ( $(\ )E: 3$ ) i

$(n + (m + k)) = (m + (n + k))$  ,! 5 ( $\Rightarrow E: 2,4$ ) i

$\omega[n] \ \& \ \omega[(m + k)]$  ,! 6 ( $\mathbb{T}E: C1.7,5$ ) i

$( (n + (m + k)) = (m + (n + k))$   
 $\Rightarrow (n + (m + k)) = (m + (k + n)) )$   
 ,! 7 ( $\forall E: P13;$   
 $(n + (m + k))$ ): C1.7,6) i

$(n + (m + k)) = (m + (n + k))$   
 $\Rightarrow (n + (m + k)) = (m + (k + n))$   
 ,! 8 ( $(\ )E: 7$ ) i

$(n + (m + k)) = (m + (k + n))$  ,! 9 ( $\Rightarrow E: 5,8$ ) i

$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow (n + (m + k)) = (m + (k + n))$   
 ,! 10 ( $\Rightarrow I: 2,9$ ) i

$( \omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow (n + (m + k)) = (m + (k + n)) )$   
 ,! 11 ( $(\ )I: 10$ ) i

$\forall n \forall m \forall k ( \omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow (n + (m + k)) = (m + (k + n)) )$   
 ! 12 ( $\forall I: 1,11$ ) i

□

! 27. i

$\vdash \forall n \forall m \forall k ( \omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$   
 $\Rightarrow ((n + m) + k) = ((k + m) + n) )$  i

**n,m,k** ,! 1 (Prem) i

$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$	,! 2 (Prem)	i
$( \ \omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$		
$\Rightarrow ((n + m) + k) = ((k + n) + m) )$	,! 3 ( $\forall E$ : P25)	i
$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$		
$\Rightarrow ((n + m) + k) = ((k + n) + m)$	,! 4 ( $(())E$ : 3)	i
$((n + m) + k) = ((k + n) + m)$	,! 5 ( $\Rightarrow E$ : 2,4)	i
$\omega[(n + m)] \ \& \ \omega[k]$	,! 6 ( $\mathbb{T}E$ : C1.7,5)	i
$( ((n + m) + k) = ((k + n) + m)$		
$\Rightarrow ((n + m) + k) = ((k + m) + n) )$	,! 7 ( $\forall E$ : P21; ( $(n + m) + k$ ): C1.7,6)	i
$((n + m) + k) = ((k + n) + m)$		
$\Rightarrow ((n + m) + k) = ((k + m) + n)$	,! 8 ( $(())E$ : 7)	i
$((n + m) + k) = ((k + m) + n)$	,! 9 ( $\Rightarrow E$ : 5,8)	i
$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow ((n + m) + k) = ((k + m) + n)$	,! 10 ( $\Rightarrow I$ : 2,9)	i
$( \ \omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow ((n + m) + k) = ((k + m) + n) )$	,! 11 ( $(())I$ : 10)	i
$\forall n \forall m \forall k ( \ \omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow ((n + m) + k) = ((k + m) + n) )$	! 12 ( $\forall I$ : 1,11)	i

□

! 28.

$\vdash \forall n \forall m \forall k ( \ \omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$		
$\Rightarrow (n + (m + k)) = (k + (m + n)) )$		i
<b>n, m, k</b>	,! 1 (Prem)	i
$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$	,! 2 (Prem)	i
$( \ \omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$		
$\Rightarrow (n + (m + k)) = (m + (k + n)) )$	,! 3 ( $\forall E$ : P26)	i
$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$		
$\Rightarrow (n + (m + k)) = (m + (k + n))$	,! 4 ( $(())E$ : 3)	i

$(n + (m + k)) = (m + (k + n))$	,! 5 ( $\Rightarrow$ E: 2,4)	i
$\omega[n] \ \& \ \omega[(m + k)]$	,! 6 ( $\mathbb{T}$ E: C1.7,5)	i
$( (n + (m + k)) = (m + (k + n))$ $\Rightarrow (n + (m + k)) = (k + (m + n)) )$	,! 7 ( $\forall$ E: P24; $(n + (m + k))$ : C1.7,6)	i
$(n + (m + k)) = (m + (k + n))$ $\Rightarrow (n + (m + k)) = (k + (m + n))$	,! 8 ( $(())$ E: 7)	i
$(n + (m + k)) = (k + (m + n))$	,! 9 ( $\Rightarrow$ E: 5,8)	i
$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow (n + (m + k)) = (k + (m + n))$	,! 10 ( $\Rightarrow$ I: 2,9)	i
$( \omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow (n + (m + k)) = (k + (m + n)) )$	,! 11 ( $(())$ I: 10)	i
$\forall n \forall m \forall k ( \omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$ $\Rightarrow (n + (m + k)) = (k + (m + n)) )$	! 12 ( $\forall$ I: 1,11)	i
$\square$		
! 29.		i
$\vdash \forall n \forall m \forall k \forall j ( \omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \omega[j]$ $\Rightarrow ((n + m) + (k + j)) = ((n + k) + (m + j)) )$		i
$n, m, k, j$	,! 1 (Prem)	i
$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \omega[j]$	,! 2 (Prem)	i
$\omega[n]$	,! 3 ( $\&$ E: 2)	i
$\omega[m]$	,! 4 ( $\&$ E: 2)	i
$\omega[k]$	,! 5 ( $\&$ E: 2)	i
$\omega[j]$	,! 6 ( $\&$ E: 2)	i
$\omega[n] \ \& \ \omega[m]$	,! 7 ( $\&$ I: 3,4)	i
$( \omega[n] \ \& \ \omega[m] \Rightarrow \omega[(n + m)] )$	,! 8 ( $\forall$ E: C1.8)	i
$\omega[n] \ \& \ \omega[m] \Rightarrow \omega[(n + m)]$	,! 9 ( $(())$ E: 8)	i
$\omega[(n + m)]$	,! 10 ( $\Rightarrow$ E: 7,9)	i
$\omega[(n + m)] \ \& \ \omega[k]$	,! 11 ( $\&$ I: 5,10)	i

$\omega[(n + m)] \ \& \ \omega[k] \ \& \ \omega[j]$  ,! 12 (&I: 6,11) i

(  $\omega[(n + m)] \ \& \ \omega[k] \ \& \ \omega[j]$   
 $\Rightarrow ((n + m) + (k + j)) = (((n + m) + k) + j)$  )  
,! 13 ( $\forall E$ : P14;  
( $n + m$ ): C1.7,7) i

$\omega[(n + m)] \ \& \ \omega[k] \ \& \ \omega[j]$   
 $\Rightarrow ((n + m) + (k + j)) = (((n + m) + k) + j)$   
,! 14 ( $()E$ : 13) i

$((n + m) + (k + j)) = (((n + m) + k) + j)$   
,! 15 ( $\Rightarrow E$ : 12,14) i

$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$  ,! 16 (&I: 5,7) i

(  $\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow ((n + m) + k) = ((n + k) + m)$  )  
,! 17 ( $\forall E$ : P19) i

$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow ((n + m) + k) = ((n + k) + m)$   
,! 18 ( $()E$ : 17) i

$((n + m) + k) = ((n + k) + m)$  ,! 19 ( $\Rightarrow E$ : 16,18) i

$((n + m) + (k + j)) = (((n + k) + m) + j)$   
,! 20 ( $=E$ : 15,19) i

$\omega[n] \ \& \ \omega[k]$  ,! 21 (&I: 3,5) i

(  $\omega[n] \ \& \ \omega[k] \Rightarrow \omega[(n + k)]$  ) ,! 22 ( $\forall E$ : C1.8) i

$\omega[n] \ \& \ \omega[k] \Rightarrow \omega[(n + k)]$  ,! 23 ( $()E$ : 22) i

$\omega[(n + k)]$  ,! 24 ( $\Rightarrow E$ : 21,23) i

$\omega[(n + k)] \ \& \ \omega[m]$  ,! 25 (&I: 4,24) i

$\omega[(n + k)] \ \& \ \omega[m] \ \& \ \omega[j]$  ,! 26 (&I: 6,25) i

(  $\omega[(n + k)] \ \& \ \omega[m] \ \& \ \omega[j]$   
 $\Rightarrow ((n + k) + (m + j)) = (((n + k) + m) + j)$  )  
,! 27 ( $\forall E$ : P14;  
( $n + k$ ): C1.7,21) i

$\omega[(n + k)] \ \& \ \omega[m] \ \& \ \omega[j]$   
 $\Rightarrow ((n + k) + (m + j)) = (((n + k) + m) + j)$   
,! 28 ( $()E$ : 27) i

$((n + k) + (m + j)) = (((n + k) + m) + j)$   
,! 29 ( $\Rightarrow E$ : 26,28) i

$((n + m) + (k + j)) = ((n + k) + (m + j))$   
,! 30 ( $=E$ : 20,29) i

$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \omega[j]$

$$\Rightarrow ((n + m) + (k + j)) = ((n + k) + (m + j))$$

, ! 31 ( $\Rightarrow$ I: 2,30) i

$$(\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \omega[j])$$

$$\Rightarrow ((n + m) + (k + j)) = ((n + k) + (m + j))$$

, ! 32 ( $(\ )$ I: 31) i

$$\forall n \forall m \forall k \forall j (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \omega[j])$$

$$\Rightarrow ((n + m) + (k + j)) = ((n + k) + (m + j))$$

! 33 ( $\forall$ I: 1,32) i

□

! P30 through P39 are propositions establish 0 as the **Additive Identity**.

P30 through P33 say that 0 is an Additive Identity,  
P34 and P35 are corollaries, and  
P36 through P39 state that 0 is the only Additive Identity. i

! 30. Strictly, an appeal to  $\omega 0$  could be avoided, since by an easy Induction it is possible to prove  $\forall n (\omega[n] \Rightarrow \omega[0])$ .

i

$$\vdash \forall n (\omega[n] \Rightarrow n = (n + 0))$$

i

$$n$$

, ! 1 (Prem) i

$$\omega[n]$$

, ! 2 (Prem) i

$$(\omega[n] \Rightarrow \exists P \mathcal{N}[n, P])$$

, ! 3 ( $\forall$ E: IV7.8) i

$$\omega[n] \Rightarrow \exists P \mathcal{N}[n, P]$$

, ! 4 ( $(\ )$ E: 3) i

$$\exists P \mathcal{N}[n, P]$$

, ! 5 ( $\Rightarrow$ E: 2,4) i

$$\mathcal{N}[n, P]$$

, ! 6 ( $\exists$ E: 5) i

$$\omega[n] \ \& \ \mathcal{N}[n, P]$$

, ! 7 ( $\&$ I: 2,6) i

$$\omega[n] \ \& \ \omega[0] \ \& \ \mathcal{N}[n, P]$$

, ! 8 ( $\&$ I:  $\omega 0$ ,7) i

$$\omega[n] \ \& \ \omega[0] \ \& \ \mathcal{N}[n, P] \ \& \ \mathcal{N}[0, \phi]$$

, ! 9 ( $\&$ I: IV3.14,8) i

$$(P \cap \phi) \equiv \phi$$

, ! 10 ( $\forall$ E: II5.30) i

$$\omega[n] \ \& \ \omega[0] \ \& \ \mathcal{N}[n, P] \ \& \ \mathcal{N}[0, \phi] \ \& \ (P \cap \phi) \equiv \phi$$

, ! 11 ( $\&$ I: 9,10) i

$$(P \cup \phi) \equiv P$$

, ! 10 ( $\forall$ E: II5.19) i

$$\omega[n] \ \& \ \mathcal{N}[n, P] \ \& \ (P \cup \phi) \equiv P$$

, ! 11 ( $\&$ I: 7,10) i

$$(\omega[n] \ \& \ \mathcal{N}[n, P] \ \& \ (P \cup \phi) \equiv P \Rightarrow \mathcal{N}[n, (P \cup \phi)])$$

, ! 12 ( $\forall$ E: IV4.6) i

$\omega[n] \ \& \ \mathfrak{N}[n,P] \ \& \ (P \cup \phi) \equiv P \Rightarrow \mathfrak{N}[n,(P \cup \phi)]$	,! 13 ((E: 12)    i
$\mathfrak{N}[n,(P \cup \phi)]$	,! 14 ( $\Rightarrow$ E: 11,13)    i
$\omega[n] \ \& \ \omega[0] \ \& \ \mathfrak{N}[n,P] \ \& \ \mathfrak{N}[0,\phi] \ \& \ (P \cap \phi) \equiv \phi \ \& \ \mathfrak{N}[n,(P \cup \phi)]$	,! 15 (&I: 11,14)    i
$( \ \omega[n] \ \& \ \omega[0] \ \& \ \mathfrak{N}[n,P] \ \& \ \mathfrak{N}[0,\phi] \ \& \ (P \cap \phi) \equiv \phi$ $\ \& \ \mathfrak{N}[n,(P \cup \phi)]$ $\Rightarrow n = (n + 0) )$	,! 16 ( $\forall$ E: C1.17)    i
$\omega[n] \ \& \ \omega[0] \ \& \ \mathfrak{N}[n,P] \ \& \ \mathfrak{N}[0,\phi] \ \& \ (P \cap \phi) \equiv \phi \ \& \ \mathfrak{N}[n,(P \cup \phi)]$ $\Rightarrow n = (n + 0)$	,! 17 ((E: 16)    i
$n = (n + 0)$	,! 18 ( $\Rightarrow$ E: 15,17)    i
$\omega[n] \Rightarrow n = (n + 0)$	,! 19 ( $\Rightarrow$ I: 2,18)    i
$( \ \omega[n] \Rightarrow n = (n + 0) )$	,! 20 ((I: 19)    i
$\forall n ( \ \omega[n] \Rightarrow n = (n + 0) )$	! 21 ( $\forall$ I: 1,20)    i
$\square$	
! 31.	i
$\vdash \forall n ( \ \omega[n] \Rightarrow n = (0 + n) )$	i
$n$	,! 1 (Prem)    i
$\omega[n]$	,! 2 (Prem)    i
$( \ \omega[n] \Rightarrow n = (n + 0) )$	,! 3 ( $\forall$ E: P30)    i
$\omega[n] \Rightarrow n = (n + 0)$	,! 4 ((E: 3)    i
$n = (n + 0)$	,! 5 ( $\Rightarrow$ E: 2,4)    i
$( \ n = (n + 0) \Rightarrow n = (0 + n) )$	,! 6 ( $\forall$ E: P7)    i
$n = (n + 0) \Rightarrow n = (0 + n)$	,! 7 ((E: 6)    i
$n = (0 + n)$	,! 8 ( $\Rightarrow$ E: 5,7)    i
$\omega[n] \Rightarrow n = (0 + n)$	,! 9 ( $\Rightarrow$ I: 2,8)    i
$( \ \omega[n] \Rightarrow n = (0 + n) )$	,! 10 ((I: 9)    i
$\forall n ( \ \omega[n] \Rightarrow n = (0 + n) )$	! 11 ( $\forall$ I: 1,10)    i
$\square$	

! 32. i

$\vdash \forall n ( \omega[n] \Rightarrow (n + 0) = n )$  i

**n** , ! 1 (Prem) i

**$\omega[n]$**  , ! 2 (Prem) i

**$( \omega[n] \Rightarrow n = (n + 0) )$**  , ! 3 ( $\forall E$ : P30) i

**$\omega[n] \Rightarrow n = (n + 0)$**  , ! 4 ( $(())E$ : 3) i

**$n = (n + 0)$**  , ! 5 ( $\Rightarrow E$ : 2,4) i

**$n = n$**  , ! 6 ( $=I$ ) i

**$(n + 0) = n$**  , ! 7 ( $=E$ : 5,6) i

**$\omega[n] \Rightarrow (n + 0) = n$**  , ! 8 ( $\Rightarrow I$ : 2,7) i

**$( \omega[n] \Rightarrow (n + 0) = n )$**  , ! 9 ( $(())I$ : 8) i

$\forall n ( \omega[n] \Rightarrow (n + 0) = n )$  ! 10 ( $\forall I$ : 1,9) i

□

! 33. i

$\vdash \forall n ( \omega[n] \Rightarrow (0 + n) = n )$  i

**n** , ! 1 (Prem) i

**$\omega[n]$**  , ! 2 (Prem) i

**$( \omega[n] \Rightarrow n = (0 + n) )$**  , ! 3 ( $\forall E$ : P31) i

**$\omega[n] \Rightarrow n = (0 + n)$**  , ! 4 ( $(())E$ : 3) i

**$n = (0 + n)$**  , ! 5 ( $\Rightarrow E$ : 2,4) i

**$n = n$**  , ! 6 ( $=I$ ) i

**$(0 + n) = n$**  , ! 7 ( $=E$ : 5,6) i

**$\omega[n] \Rightarrow (0 + n) = n$**  , ! 8 ( $\Rightarrow I$ : 2,7) i

**$( \omega[n] \Rightarrow (0 + n) = n )$**  , ! 9 ( $(())I$ : 8) i

$\forall n ( \omega[n] \Rightarrow (0 + n) = n )$  ! 10 ( $\forall I$ : 1,9) i

□

! 34. i

$\vdash \forall n \forall m ( (n + 0) = m \Rightarrow n = m )$  i

<b>n, m</b>	,! 1 (Prem)	i
$(n + 0) = m$	,! 2 (Prem)	i
$\omega[n] \ \& \ \omega[0]$	,! 3 ( <b>TE</b> : C1.7,2)	i
$\omega[n]$	,! 4 (&E: 3)	i
$(\omega[n] \Rightarrow (n + 0) = n)$	,! 5 ( $\forall$ E: P32)	i
$\omega[n] \Rightarrow (n + 0) = n$	,! 6 (()E: 5)	i
$(n + 0) = n$	,! 7 ( $\Rightarrow$ E: 4,6)	i
$n = m$	,! 8 (=E: 2,7)	i
$(n + 0) = m \Rightarrow n = m$	,! 9 ( $\Rightarrow$ I: 2,8)	i
$( (n + 0) = m \Rightarrow n = m )$	,! 10 (()I: 9)	i
$\forall n \forall m ( (n + 0) = m \Rightarrow n = m )$	! 11 ( $\forall$ I: 1,10)	i
$\square$		

! 35.

$\vdash \forall n \forall m ( n = (m + 0) \Rightarrow n = m )$		i
<b>n, m</b>	,! 1 (Prem)	i
$n = (m + 0)$	,! 2 (Prem)	i
$\omega[m] \ \& \ \omega[0]$	,! 3 ( <b>TE</b> : C1.7,2)	i
$\omega[m]$	,! 4 (&E: 3)	i
$(\omega[m] \Rightarrow (m + 0) = m)$	,! 5 ( $\forall$ E: P32)	i
$\omega[m] \Rightarrow (m + 0) = m$	,! 6 (()E: 5)	i
$(m + 0) = m$	,! 7 ( $\Rightarrow$ E: 4,6)	i
$n = m$	,! 8 (=E: 2,7)	i
$n = (m + 0) \Rightarrow n = m$	,! 9 ( $\Rightarrow$ I: 2,8)	i
$( n = (m + 0) \Rightarrow n = m )$	,! 10 (()I: 9)	i
$\forall n \forall m ( n = (m + 0) \Rightarrow n = m )$	! 11 ( $\forall$ I: 1,10)	i
$\square$		

! 36.

$\vdash \forall n \forall m ( n = (n + m) \Rightarrow m = 0 )$		i
<b>n, m</b>	,! 1 (Prem)	i

$$\begin{array}{lll} \mathbf{n} = (\mathbf{n} + \mathbf{m}) & ,! 2 \text{ (Prem)} & i \\ \omega[\mathbf{n}] \ \& \ \omega[\mathbf{m}] & ,! 3 \text{ (TE: C1.7,2)} \quad i \\ \omega[\mathbf{n}] & ,! 4 \text{ (&E: 3)} & i \\ (\mathbf{n} = (\mathbf{n} + \mathbf{m}) & & \\ \Rightarrow \exists \mathbf{A} \exists \mathbf{B} (\mathfrak{N}[\mathbf{n}, \mathbf{A}] \ \& \ \mathfrak{N}[\mathbf{m}, \mathbf{B}] \ \& \ (\mathbf{A} \cap \mathbf{B}) \equiv \phi \ \& \ \mathfrak{N}[\mathbf{n}, (\mathbf{A} \cup \mathbf{B})]) & ) \\ & ,! 5 \text{ (\forall E: C1.19)} & i \\ \mathbf{n} = (\mathbf{n} + \mathbf{m}) & & \\ \Rightarrow \exists \mathbf{A} \exists \mathbf{B} (\mathfrak{N}[\mathbf{n}, \mathbf{A}] \ \& \ \mathfrak{N}[\mathbf{m}, \mathbf{B}] \ \& \ (\mathbf{A} \cap \mathbf{B}) \equiv \phi \ \& \ \mathfrak{N}[\mathbf{n}, (\mathbf{A} \cup \mathbf{B})]) & \\ & ,! 6 \text{ (()E: 5)} & i \\ \exists \mathbf{A} \exists \mathbf{B} (\mathfrak{N}[\mathbf{n}, \mathbf{A}] \ \& \ \mathfrak{N}[\mathbf{m}, \mathbf{B}] \ \& \ (\mathbf{A} \cap \mathbf{B}) \equiv \phi \ \& \ \mathfrak{N}[\mathbf{n}, (\mathbf{A} \cup \mathbf{B})]) & \\ & ,! 7 \text{ (\Rightarrow E: 2,6)} & i \\ \exists \mathbf{B} (\mathfrak{N}[\mathbf{n}, \mathbf{A}] \ \& \ \mathfrak{N}[\mathbf{m}, \mathbf{B}] \ \& \ (\mathbf{A} \cap \mathbf{B}) \equiv \phi \ \& \ \mathfrak{N}[\mathbf{n}, (\mathbf{A} \cup \mathbf{B})]) & \\ & ,! 8 \text{ (\exists E: 7)} & i \\ (\mathfrak{N}[\mathbf{n}, \mathbf{A}] \ \& \ \mathfrak{N}[\mathbf{m}, \mathbf{B}] \ \& \ (\mathbf{A} \cap \mathbf{B}) \equiv \phi \ \& \ \mathfrak{N}[\mathbf{n}, (\mathbf{A} \cup \mathbf{B})]) & \\ & ,! 9 \text{ (\exists E: 8)} & i \\ \mathfrak{N}[\mathbf{n}, \mathbf{A}] \ \& \ \mathfrak{N}[\mathbf{m}, \mathbf{B}] \ \& \ (\mathbf{A} \cap \mathbf{B}) \equiv \phi \ \& \ \mathfrak{N}[\mathbf{n}, (\mathbf{A} \cup \mathbf{B})] & \\ & ,! 10 \text{ (()E: 9)} & i \\ \mathfrak{N}[\mathbf{n}, \mathbf{A}] & ,! 11 \text{ (&E: 10)} & i \\ \mathfrak{N}[\mathbf{m}, \mathbf{B}] & ,! 12 \text{ (&E: 10)} & i \\ (\mathbf{A} \cap \mathbf{B}) \equiv \phi & ,! 13 \text{ (&E: 10)} & i \\ \mathfrak{N}[\mathbf{n}, (\mathbf{A} \cup \mathbf{B})] & ,! 14 \text{ (&E: 10)} & i \\ \omega[\mathbf{n}] \ \& \ \mathfrak{N}[\mathbf{n}, \mathbf{A}] & ,! 15 \text{ (&I: 4,11)} \quad i \\ \omega[\mathbf{n}] \ \& \ \mathfrak{N}[\mathbf{n}, \mathbf{A}] \ \& \ \mathfrak{N}[\mathbf{n}, (\mathbf{A} \cup \mathbf{B})] & ,! 16 \text{ (&I: 14,15)} \quad i \end{array}$$

! Lines 17 to 36 show that  $\forall x \neg \mathbf{B}[x]$ , in order to prepare the appeal to IV3.16. i

$$\begin{array}{lll} \mathbf{A} \subseteq (\mathbf{A} \cup \mathbf{B}) & ,! 17 \text{ (\forall E: II2.12)} & i \\ \omega[\mathbf{n}] \ \& \ \mathfrak{N}[\mathbf{n}, \mathbf{A}] \ \& \ \mathfrak{N}[\mathbf{n}, (\mathbf{A} \cup \mathbf{B})] \ \& \ \mathbf{A} \subseteq (\mathbf{A} \cup \mathbf{B}) & \\ & ,! 18 \text{ (&I: 16,17)} & i \\ (\omega[\mathbf{n}] \ \& \ \mathfrak{N}[\mathbf{n}, \mathbf{A}] \ \& \ \mathfrak{N}[\mathbf{n}, (\mathbf{A} \cup \mathbf{B})] \ \& \ \mathbf{A} \subseteq (\mathbf{A} \cup \mathbf{B})) & \\ \Rightarrow \mathbf{A} \equiv (\mathbf{A} \cup \mathbf{B}) & & \\ & ,! 19 \text{ (\forall E: IV4.18)} & i \\ \omega[\mathbf{n}] \ \& \ \mathfrak{N}[\mathbf{n}, \mathbf{A}] \ \& \ \mathfrak{N}[\mathbf{n}, (\mathbf{A} \cup \mathbf{B})] \ \& \ \mathbf{A} \subseteq (\mathbf{A} \cup \mathbf{B}) \Rightarrow \mathbf{A} \equiv (\mathbf{A} \cup \mathbf{B}) & \\ & ,! 20 \text{ (()E: 19)} & i \end{array}$$

$\mathbf{A} \equiv (\mathbf{A} \cup \mathbf{B})$	,! 21 ( $\Rightarrow$ E: 18,20) ;
$(\mathbf{A} \equiv (\mathbf{A} \cup \mathbf{B}) \Rightarrow (\mathbf{A} \cup \mathbf{B}) \equiv \mathbf{A})$	,! 22 ( $\forall$ E: III.1.10) ;
$\mathbf{A} \equiv (\mathbf{A} \cup \mathbf{B}) \Rightarrow (\mathbf{A} \cup \mathbf{B}) \equiv \mathbf{A}$	,! 23 ( $(\ )$ E: 22) ;
$(\mathbf{A} \cup \mathbf{B}) \equiv \mathbf{A}$	,! 24 ( $\Rightarrow$ E: 21,23) ;
$(\mathbf{A} \cap \mathbf{B}) \equiv \phi \ \& \ (\mathbf{A} \cup \mathbf{B}) \equiv \mathbf{A}$	,! 25 ( $\&$ I: 13,24) ;
$( (\mathbf{A} \cap \mathbf{B}) \equiv \phi \ \& \ (\mathbf{A} \cup \mathbf{B}) \equiv \mathbf{A} \Rightarrow (\mathbf{A} \setminus \mathbf{A}) \equiv \mathbf{B} )$	,! 26 ( $\forall$ E: II7.78) ;
$(\mathbf{A} \cap \mathbf{B}) \equiv \phi \ \& \ (\mathbf{A} \cup \mathbf{B}) \equiv \mathbf{A} \Rightarrow (\mathbf{A} \setminus \mathbf{A}) \equiv \mathbf{B}$	,! 27 ( $(\ )$ E: 26) ;
$(\mathbf{A} \setminus \mathbf{A}) \equiv \mathbf{B}$	,! 28 ( $\Rightarrow$ E: 25,27) ;
$(\mathbf{A} \setminus \mathbf{A}) \equiv \phi$	,! 29 ( $\forall$ E: II7.51) ;
$(\mathbf{A} \setminus \mathbf{A}) \equiv \mathbf{B} \ \& \ (\mathbf{A} \setminus \mathbf{A}) \equiv \phi$	,! 30 ( $\&$ I: 28,29) ;
$( (\mathbf{A} \setminus \mathbf{A}) \equiv \mathbf{B} \ \& \ (\mathbf{A} \setminus \mathbf{A}) \equiv \phi \Rightarrow \mathbf{B} \equiv \phi )$	,! 31 ( $\forall$ E: III.1.19) ;
$(\mathbf{A} \setminus \mathbf{A}) \equiv \mathbf{B} \ \& \ (\mathbf{A} \setminus \mathbf{A}) \equiv \phi \Rightarrow \mathbf{B} \equiv \phi$	,! 32 ( $(\ )$ E: 31) ;
$\mathbf{B} \equiv \phi$	,! 33 ( $\Rightarrow$ E: 30,32) ;
$(\mathbf{B} \equiv \phi \Rightarrow \forall x \neg \mathbf{B}[x])$	,! 34 ( $\forall$ E: II5.5) ;
$\mathbf{B} \equiv \phi \Rightarrow \forall x \neg \mathbf{B}[x]$	,! 35 ( $(\ )$ E: 34) ;
$\forall x \neg \mathbf{B}[x]$	,! 36 ( $\Rightarrow$ E: 33,35) ;
$\mathfrak{N}[\mathbf{m}, \mathbf{B}] \ \& \ \forall x \neg \mathbf{B}[x]$	,! 37 ( $\&$ I: 12,36) ;
$(\mathfrak{N}[\mathbf{m}, \mathbf{B}] \ \& \ \forall x \neg \mathbf{B}[x] \Rightarrow \mathbf{m} = 0)$	,! 38 ( $\forall$ E: IV3.16) ;
$\mathfrak{N}[\mathbf{m}, \mathbf{B}] \ \& \ \forall x \neg \mathbf{B}[x] \Rightarrow \mathbf{m} = 0$	,! 39 ( $(\ )$ E: 38) ;
$\mathbf{m} = 0$	,! 40 ( $\Rightarrow$ E: 37,39) ;
$\mathbf{n} = (\mathbf{n} + \mathbf{m}) \Rightarrow \mathbf{m} = 0$	,! 41 ( $\Rightarrow$ I: 2,40) ;
$(\mathbf{n} = (\mathbf{n} + \mathbf{m}) \Rightarrow \mathbf{m} = 0)$	,! 42 ( $(\ )$ I: 41) ;
$\forall n \forall m (\mathbf{n} = (\mathbf{n} + \mathbf{m}) \Rightarrow \mathbf{m} = 0)$	,! 43 ( $\forall$ I: 1,42) ;
$\square$	
! 37.	; i
$\vdash \forall n \forall m (\mathbf{n} = (\mathbf{m} + \mathbf{n}) \Rightarrow \mathbf{m} = 0)$	; i

$n, m$	,! 1 (Prem)	i
$n = (m + n)$	,! 2 (Prem)	i
$(n = (m + n) \Rightarrow n = (n + m))$	,! 3 ( $\forall E$ : P7)	i
$n = (m + n) \Rightarrow n = (n + m)$	,! 4 ( $(\Rightarrow)E$ : 3)	i
$n = (n + m)$	,! 5 ( $\Rightarrow E$ : 2,4)	i
$(n = (n + m) \Rightarrow m = 0)$	,! 6 ( $\forall E$ : P36)	i
$n = (n + m) \Rightarrow m = 0$	,! 7 ( $(\Rightarrow)E$ : 6)	i
$m = 0$	,! 8 ( $\Rightarrow E$ : 5,7)	i
$n = (m + n) \Rightarrow m = 0$	,! 9 ( $\Rightarrow I$ : 2,8)	i
$(n = (m + n) \Rightarrow m = 0)$	,! 10 ( $(\Rightarrow)I$ : 9)	i
$\forall n \forall m (n = (m + n) \Rightarrow m = 0)$	,! 11 ( $\forall I$ : 1,10)	i

□

! 38.

$\vdash \forall n \forall m ((n + m) = n \Rightarrow m = 0)$		i
$n, m$	,! 1 (Prem)	i
$(n + m) = n$	,! 2 (Prem)	i
$n = n$	,! 3 ( $=I$ )	i
$n = (n + m)$	,! 4 ( $=E$ : 2,3)	i
$(n = (n + m) \Rightarrow m = 0)$	,! 5 ( $\forall E$ : P36)	i
$n = (n + m) \Rightarrow m = 0$	,! 6 ( $(\Rightarrow)E$ : 5)	i
$m = 0$	,! 7 ( $\Rightarrow E$ : 4,6)	i
$(n + m) = n \Rightarrow m = 0$	,! 8 ( $\Rightarrow I$ : 2,7)	i
$((n + m) = n \Rightarrow m = 0)$	,! 9 ( $(\Rightarrow)I$ : 8)	i
$\forall n \forall m ((n + m) = n \Rightarrow m = 0)$	! 10 ( $\forall I$ : 1,9)	i

□

! 39.

$\vdash \forall n \forall m ((m + n) = n \Rightarrow m = 0)$		i
$n, m$	,! 1 (Prem)	i

$(m + n) = n$	,! 2 (Prem)	i
$n = n$	,! 3 (=I)	i
$n = (m + n)$	,! 4 (=E: 2,3)	i
$(n = (m + n) \Rightarrow m = 0)$	,! 5 ( $\forall$ E: P37)	i
$n = (m + n) \Rightarrow m = 0$	,! 6 (()E: 5)	i
$m = 0$	,! 7 ( $\Rightarrow$ E: 4,6)	i
$(m + n) = n \Rightarrow m = 0$	,! 8 ( $\Rightarrow$ I: 2,7)	i
$( (m + n) = n \Rightarrow m = 0 )$	,! 9 (()I: 8)	i
$\forall n \forall m ( (m + n) = n \Rightarrow m = 0 )$	! 10 ( $\forall$ I: 1,9)	i

□

! P40 through P48 relate addition by 1 with succession. i

! 40. i

$\vdash \forall n ( \omega[n] \Rightarrow (n') = (n + 1) )$  i

$n$  ,! 1 (Prem) i

$\omega[n]$  ,! 2 (Prem) i

$\omega[n] \ \& \ \omega[1]$  ,! 3 (&I: IV9.2,2) i

$( \omega[n] \Rightarrow \omega[(n')] )$  ,! 4 ( $\forall$ E: IV8.25) i

$\omega[n] \Rightarrow \omega[(n')]$  ,! 5 (()E: 4) i

$\omega[(n')]$  ,! 6 ( $\Rightarrow$ E: 2,5) i

$\omega[n] \ \& \ \omega[1] \ \& \ \omega[(n')]$  ,! 7 (&I: 3,6) i

$( \omega[n] \Rightarrow \exists P \exists a ( \mathcal{N}[n,P] \ \& \ \neg P[a] \ \& \ \mathcal{N}[(n'), (P \cup (a^\bullet))] ) )$   
, ! 8 ( $\forall$ E: IV8.23) i

$\omega[n] \Rightarrow \exists P \exists a ( \mathcal{N}[n,P] \ \& \ \neg P[a] \ \& \ \mathcal{N}[(n'), (P \cup (a^\bullet))] )$   
, ! 9 (()E: 8) i

$\exists P \exists a ( \mathcal{N}[n,P] \ \& \ \neg P[a] \ \& \ \mathcal{N}[(n'), (P \cup (a^\bullet))] )$   
, ! 10 ( $\Rightarrow$ E: 2,9) i

$\exists a ( \mathcal{N}[n,P] \ \& \ \neg P[a] \ \& \ \mathcal{N}[(n'), (P \cup (a^\bullet))] )$   
, ! 11 ( $\exists$ E: 10) i

$( \mathcal{N}[n,P] \ \& \ \neg P[a] \ \& \ \mathcal{N}[(n'), (P \cup (a^\bullet))] )$   
, ! 12 ( $\exists$ E: 11) i

$\mathcal{N}[\mathbf{n}, \mathbf{P}] \ \& \ \neg \mathbf{P}[\mathbf{a}] \ \& \ \mathcal{N}[(\mathbf{n}'), (\mathbf{P} \cup (\mathbf{a}^\bullet))]$	,! 13 ((E: 12)	i
$\mathcal{N}[\mathbf{n}, \mathbf{P}]$	,! 14 (&E: 13)	i
$\neg \mathbf{P}[\mathbf{a}]$	,! 15 (&E: 13)	i
$\mathcal{N}[(\mathbf{n}'), (\mathbf{P} \cup (\mathbf{a}^\bullet))]$	,! 16 (&E: 13)	i
$\omega[\mathbf{n}] \ \& \ \omega[1] \ \& \ \omega[(\mathbf{n}')] \ \& \ \mathcal{N}[\mathbf{n}, \mathbf{P}]$	,! 17 (&I: 7,14)	i
$\mathcal{N}[1, (\mathbf{a}^\bullet)]$	,! 18 ( $\forall$ E: IV9.3)	i
$\omega[\mathbf{n}] \ \& \ \omega[1] \ \& \ \omega[(\mathbf{n}')] \ \& \ \mathcal{N}[\mathbf{n}, \mathbf{P}] \ \& \ \mathcal{N}[1, (\mathbf{a}^\bullet)]$	,! 19 (&I: 17,18)	i
$( \neg \mathbf{P}[\mathbf{a}] \Rightarrow (\mathbf{P} \cap (\mathbf{a}^\bullet)) \equiv \phi )$	,! 20 ( $\forall$ E: II8.40)	i
$\neg \mathbf{P}[\mathbf{a}] \Rightarrow (\mathbf{P} \cap (\mathbf{a}^\bullet)) \equiv \phi$	,! 21 ((E: 20)	i
$(\mathbf{P} \cap (\mathbf{a}^\bullet)) \equiv \phi$	,! 22 ( $\Rightarrow$ E: 15,21)	i
$\omega[\mathbf{n}] \ \& \ \omega[1] \ \& \ \omega[(\mathbf{n}')] \ \& \ \mathcal{N}[\mathbf{n}, \mathbf{P}] \ \& \ \mathcal{N}[1, (\mathbf{a}^\bullet)] \ \& \ (\mathbf{P} \cap (\mathbf{a}^\bullet)) \equiv \phi$	,! 23 (&I: 19,22)	i
$\omega[\mathbf{n}] \ \& \ \omega[1] \ \& \ \omega[(\mathbf{n}')] \ \& \ \mathcal{N}[\mathbf{n}, \mathbf{P}] \ \& \ \mathcal{N}[1, (\mathbf{a}^\bullet)] \ \& \ (\mathbf{P} \cap (\mathbf{a}^\bullet)) \equiv \phi$ $\ \& \ \mathcal{N}[(\mathbf{n}'), (\mathbf{P} \cup (\mathbf{a}^\bullet))]$	,! 24 (&I: 16,23)	i
$( \omega[\mathbf{n}] \ \& \ \omega[1] \ \& \ \omega[(\mathbf{n}')] \ \& \ \mathcal{N}[\mathbf{n}, \mathbf{P}] \ \& \ \mathcal{N}[1, (\mathbf{a}^\bullet)]$ $\ \& \ (\mathbf{P} \cap (\mathbf{a}^\bullet)) \equiv \phi \ \& \ \mathcal{N}[(\mathbf{n}'), (\mathbf{P} \cup (\mathbf{a}^\bullet))]$ $\Rightarrow (\mathbf{n}') = (\mathbf{n} + 1) )$	,! 25 ( $\forall$ E: C1.17; ( $\mathbf{n}'$ ): IV8.12,2)	i
$\omega[\mathbf{n}] \ \& \ \omega[1] \ \& \ \omega[(\mathbf{n}')] \ \& \ \mathcal{N}[\mathbf{n}, \mathbf{P}] \ \& \ \mathcal{N}[1, (\mathbf{a}^\bullet)]$ $\ \& \ (\mathbf{P} \cap (\mathbf{a}^\bullet)) \equiv \phi \ \& \ \mathcal{N}[(\mathbf{n}'), (\mathbf{P} \cup (\mathbf{a}^\bullet))]$ $\Rightarrow (\mathbf{n}') = (\mathbf{n} + 1)$	,! 26 ((E: 25)	i
$(\mathbf{n}') = (\mathbf{n} + 1)$	,! 27 ( $\Rightarrow$ E: 24,26)	i
$\omega[\mathbf{n}] \Rightarrow (\mathbf{n}') = (\mathbf{n} + 1)$	,! 28 ( $\Rightarrow$ I: 2,27)	i
$( \omega[\mathbf{n}] \Rightarrow (\mathbf{n}') = (\mathbf{n} + 1) )$	,! 29 ((I: 28)	i
$\forall \mathbf{n} ( \omega[\mathbf{n}] \Rightarrow (\mathbf{n}') = (\mathbf{n} + 1) )$	! 30 ( $\forall$ I: 1,29)	i
$\square$		
<b>! 41.</b>		i
$\vdash \forall \mathbf{n} ( \omega[\mathbf{n}] \Rightarrow (\mathbf{n}') = (1 + \mathbf{n}) )$		i

<b>n</b>	,! 1 (Prem)	i
$\omega[n]$	,! 2 (Prem)	i
$(\omega[n] \Rightarrow (n') = (n + 1))$	,! 3 ( $\forall E$ : P40)	i
$\omega[n] \Rightarrow (n') = (n + 1)$	,! 4 ( $(\Rightarrow)E$ : 3)	i
$(n') = (n + 1)$	,! 5 ( $\Rightarrow E$ : 2,4)	i
$((n') = (n + 1) \Rightarrow (n') = (1 + n))$	,! 6 ( $\forall E$ : P7; ( $n'$ ): IV8.12,2)	i
$(n') = (n + 1) \Rightarrow (n') = (1 + n)$	,! 7 ( $(\Rightarrow)E$ : 6)	i
$(n') = (1 + n)$	,! 8 ( $\Rightarrow E$ : 2,7)	i
$\omega[n] \Rightarrow (n') = (1 + n)$	,! 9 ( $\Rightarrow I$ : 2,8)	i
$(\omega[n] \Rightarrow (n') = (1 + n))$	,! 10 ( $(\Rightarrow)I$ : 9)	i
$\forall n (\omega[n] \Rightarrow (n') = (1 + n))$	! 11 ( $\forall I$ : 1,10)	i

□

! 42.

$\vdash \forall n (\omega[n] \Rightarrow (n + 1) = (n'))$		i
<b>n</b>	,! 1 (Prem)	i
$\omega[n]$	,! 2 (Prem)	i
$(\omega[n] \Rightarrow (n') = (n + 1))$	,! 3 ( $\forall E$ : P40)	i
$\omega[n] \Rightarrow (n') = (n + 1)$	,! 4 ( $(\Rightarrow)E$ : 3)	i
$(n') = (n + 1)$	,! 5 ( $\Rightarrow E$ : 2,4)	i
$(n') = (n')$	,! 6 ( $=E$ : 5,5)	i
$(n + 1) = (n')$	,! 7 ( $=E$ : 5,6)	i
$\omega[n] \Rightarrow (n + 1) = (n')$	,! 8 ( $\Rightarrow I$ : 2,7)	i
$(\omega[n] \Rightarrow (n + 1) = (n'))$	,! 9 ( $(\Rightarrow)I$ : 8)	i
$\forall n (\omega[n] \Rightarrow (n + 1) = (n'))$	! 10 ( $\forall I$ : 1,9)	i

□

! 43.

$\vdash \forall n (\omega[n] \Rightarrow (1 + n) = (n'))$		i
<b>n</b>	,! 1 (Prem)	i

$\omega[\mathbf{n}]$	,! 2 (Prem)	i
$( \omega[\mathbf{n}] \Rightarrow (\mathbf{n}') = (1 + \mathbf{n}) )$	,! 3 ( $\forall\text{E}$ : P41)	i
$\omega[\mathbf{n}] \Rightarrow (\mathbf{n}') = (1 + \mathbf{n})$	,! 4 ( $(\ )\text{E}$ : 3)	i
$(\mathbf{n}') = (1 + \mathbf{n})$	,! 5 ( $\Rightarrow\text{E}$ : 2,4)	i
$(\mathbf{n}') = (\mathbf{n}')$	,! 6 ( $=\text{E}$ : 5,5)	i
$(1 + \mathbf{n}) = (\mathbf{n}')$	,! 7 ( $=\text{E}$ : 5,6)	i
$\omega[\mathbf{n}] \Rightarrow (1 + \mathbf{n}) = (\mathbf{n}')$	,! 8 ( $\Rightarrow\text{I}$ : 2,7)	i
$( \omega[\mathbf{n}] \Rightarrow (1 + \mathbf{n}) = (\mathbf{n}') )$	,! 9 ( $(\ )\text{I}$ : 8)	i
$\forall \mathbf{n} ( \omega[\mathbf{n}] \Rightarrow (1 + \mathbf{n}) = (\mathbf{n}') )$	! 10 ( $\forall\text{I}$ : 1,9)	i

□

! 44.

$\vdash \forall \mathbf{n} \forall \mathbf{m} ( (\mathbf{n}') = \mathbf{m} \Rightarrow (\mathbf{n} + 1) = \mathbf{m} )$		i
$\mathbf{n}, \mathbf{m}$	,! 1 (Prem)	i
$(\mathbf{n}') = \mathbf{m}$	,! 2 (Prem)	i
$\omega[\mathbf{n}]$	,! 3 ( $\mathbb{D}\text{P}$ : IV8.12,2)	i
$( \omega[\mathbf{n}] \Rightarrow (\mathbf{n} + 1) = (\mathbf{n}') )$	,! 4 ( $\forall\text{E}$ : P42)	i
$\omega[\mathbf{n}] \Rightarrow (\mathbf{n} + 1) = (\mathbf{n}')$	,! 5 ( $(\ )\text{E}$ : 4)	i
$(\mathbf{n} + 1) = (\mathbf{n}')$	,! 6 ( $\Rightarrow\text{E}$ : 3,5)	i
$(\mathbf{n} + 1) = \mathbf{m}$	,! 7 ( $=\text{E}$ : 2,6)	i
$(\mathbf{n}') = \mathbf{m} \Rightarrow (\mathbf{n} + 1) = \mathbf{m}$	,! 8 ( $\Rightarrow\text{I}$ : 2,7)	i
$( (\mathbf{n}') = \mathbf{m} \Rightarrow (\mathbf{n} + 1) = \mathbf{m} )$	,! 9 ( $(\ )\text{I}$ : 8)	i
$\forall \mathbf{n} \forall \mathbf{m} ( (\mathbf{n}') = \mathbf{m} \Rightarrow (\mathbf{n} + 1) = \mathbf{m} )$	! 10 ( $\forall\text{I}$ : 1,9)	i

□

! 45.

$\vdash \forall \mathbf{n} \forall \mathbf{m} ( (\mathbf{n} + 1) = \mathbf{m} \Rightarrow (\mathbf{n}') = \mathbf{m} )$		i
$\mathbf{n}, \mathbf{m}$	,! 1 (Prem)	i
$(\mathbf{n} + 1) = \mathbf{m}$	,! 2 (Prem)	i
$\omega[\mathbf{n}] \ \& \ \omega[1]$	,! 3 ( $\mathbb{T}\text{E}$ : C1.7,2)	i

$\omega[n]$	,! 4 (&E: 3)	i
$(\omega[n] \Rightarrow (n') = (n + 1))$	,! 5 ( $\forall$ E: P40)	i
$\omega[n] \Rightarrow (n') = (n + 1)$	,! 6 (()E: 5)	i
$(n') = (n + 1)$	,! 7 ( $\Rightarrow$ E: 4,6)	i
$(n') = m$	,! 8 (=E: 2,7)	i
$(n + 1) = m \Rightarrow (n') = m$	,! 9 ( $\Rightarrow$ I: 2,8)	i
$( (n + 1) = m \Rightarrow (n') = m )$	,! 10 (()I: 9)	i
$\forall n \forall m ( (n + 1) = m \Rightarrow (n') = m )$	! 11 ( $\forall$ I: 1,10)	i
$\square$		

! 46.

$\vdash \forall n \forall m ( (n') = m \Leftrightarrow (n + 1) = m )$		i
$n, m$	,! 1 (Prem)	i
$( (n') = m \Rightarrow (n + 1) = m )$	,! 2 ( $\forall$ E: P44)	i
$(n') = m \Rightarrow (n + 1) = m$	,! 3 (()E: 2)	i
$( (n + 1) = m \Rightarrow (n') = m )$	,! 4 ( $\forall$ E: P45)	i
$(n + 1) = m \Rightarrow (n') = m$	,! 5 (()E: 4)	i
$(n') = m \Leftrightarrow (n + 1) = m$	,! 6 ( $\Leftrightarrow$ I: 3,5)	i
$( (n') = m \Leftrightarrow (n + 1) = m )$	,! 7 (()I: 6)	i
$\forall n \forall m ( (n') = m \Leftrightarrow (n + 1) = m )$	! 8 ( $\forall$ I: 1,7)	i
$\square$		

! 47.

$\vdash \forall n \forall m ( \omega[n] \ \& \ \sigma[n,m] \Rightarrow (n + 1) = m )$		i
$n, m$	,! 1 (Prem)	i
$\omega[n] \ \& \ \sigma[n,m]$	,! 2 (Prem)	i
$( \omega[n] \ \& \ \sigma[n,m] \Rightarrow (n') = m )$	,! 3 ( $\forall$ E: IV8.18)	i
$\omega[n] \ \& \ \sigma[n,m] \Rightarrow (n') = m$	,! 4 (()E: 3)	i
$(n') = m$	,! 5 ( $\Rightarrow$ E: 2,4)	i
$( (n') = m \Rightarrow (n + 1) = m )$	,! 6 ( $\forall$ E: P44)	i

$(n') = m \Rightarrow (n + 1) = m$	, ! 7 (( )E: 6)	i
$(n + 1) = m$	, ! 8 ( $\Rightarrow$ E: 5,7)	i
$\omega[n] \ \& \ \sigma[n,m] \Rightarrow (n + 1) = m$	, ! 9 ( $\Rightarrow$ I: 2,8)	i
$( \ \omega[n] \ \& \ \sigma[n,m] \Rightarrow (n + 1) = m )$	, ! 10 (( )I: 9)	i
$\forall n \forall m ( \ \omega[n] \ \& \ \sigma[n,m] \Rightarrow (n + 1) = m )$	! 11 ( $\forall$ I: 1,10)	i
$\square$		

! 48.

$\vdash \forall n \forall m ( \ \omega[n] \ \& \ \sigma[n,m] \Rightarrow m = (n + 1) )$		i
$n, m$	, ! 1 (Prem)	i
$\omega[n] \ \& \ \sigma[n,m]$	, ! 2 (Prem)	i
$( \ \omega[n] \ \& \ \sigma[n,m] \Rightarrow (n + 1) = m )$	, ! 3 ( $\forall$ E: P47)	i
$\omega[n] \ \& \ \sigma[n,m] \Rightarrow (n + 1) = m$	, ! 4 (( )E: 3)	i
$(n + 1) = m$	, ! 5 ( $\Rightarrow$ E: 2,4)	i
$m = m$	, ! 6 (=I)	i
$m = (n + 1)$	, ! 7 (=E: 5,6)	i
$\omega[n] \ \& \ \sigma[n,m] \Rightarrow m = (n + 1)$	, ! 8 ( $\Rightarrow$ I: 2,7)	i
$( \ \omega[n] \ \& \ \sigma[n,m] \Rightarrow m = (n + 1) )$	, ! 9 (( )I: 8)	i
$\forall n \forall m ( \ \omega[n] \ \& \ \sigma[n,m] \Rightarrow m = (n + 1) )$	! 10 ( $\forall$ I: 1,9)	i
$\square$		

! 49. P49 is yet another way of saying that every non-zero finite number has a finite predecessor.

$\vdash \forall n ( \ \omega[n] \ \& \ \neg n = 0 \Rightarrow \exists m (m + 1) = n )$		i
$n$	, ! 1 (Prem)	i
$\omega[n] \ \& \ \neg n = 0$	, ! 2 (Prem)	i
$( \ \omega[n] \ \& \ \neg n = 0 \Rightarrow \exists m (m') = n )$	, ! 3 ( $\forall$ E: IV8.34)	i
$\omega[n] \ \& \ \neg n = 0 \Rightarrow \exists m (m') = n$	, ! 4 (( )E: 3)	i
$\exists m (m') = n$	, ! 5 ( $\Rightarrow$ E: 2,4)	i
$(m') = n$	, ! 6 ( $\exists$ E: 5)	i
$( (m') = n \Rightarrow (m + 1) = n )$	, ! 7 ( $\forall$ E: P44)	i

$(m') = n \Rightarrow (m + 1) = n$	,! 8 (( )E: 7)	i
$(m + 1) = n$	,! 9 ( $\Rightarrow$ E: 6,8)	i
$\exists m (m + 1) = n$	,! 10 ( $\exists$ I: 9)	i
$\omega[n] \ \& \ \neg n = 0 \Rightarrow \exists m (m + 1) = n$	,! 11 ( $\Rightarrow$ I: 2,10)	i
$( \omega[n] \ \& \ \neg n = 0 \Rightarrow \exists m (m + 1) = n )$	,! 12 (( )I: 11)	i
$\forall n ( \omega[n] \ \& \ \neg n = 0 \Rightarrow \exists m (m + 1) = n )$	! 13 ( $\forall$ I: 1,12)	i
$\square$		
! 50. P50 is a corollary to P49.		i
$\vdash \forall n ( \omega[n] \Rightarrow n = 0 \vee \exists m (m + 1) = n )$		i
$n$	,! 1 (Prem)	i
$\omega[n]$	,! 2 (Prem)	i
$( n = 0 \vee \neg n = 0 )$	,! 3 ( $\forall$ E: I3.4)	i
$n = 0 \vee \neg n = 0$	,! 4 (( )E: 3)	i
$n = 0$	,! 5 (Prem)	i
$n = 0 \vee \exists m (m + 1) = n$	,! 6 ( $\vee$ I: 5)	i
$n = 0 \Rightarrow n = 0 \vee \exists m (m + 1) = n$	,! 7 ( $\Rightarrow$ I: 5,6)	i
$\neg n = 0$	,! 8 (Prem)	i
$\omega[n] \ \& \ \neg n = 0$	,! 9 ( $\&$ I: 2,8)	i
$( \omega[n] \ \& \ \neg n = 0 \Rightarrow \exists m (m + 1) = n )$	,! 10 ( $\forall$ E: P49)	i
$\omega[n] \ \& \ \neg n = 0 \Rightarrow \exists m (m + 1) = n$	,! 11 (( )E: 10)	i
$\exists m (m + 1) = n$	,! 12 ( $\Rightarrow$ E: 9,11)	i
$n = 0 \vee \exists m (m + 1) = n$	,! 13 ( $\vee$ I: 12)	i
$\neg n = 0 \Rightarrow n = 0 \vee \exists m (m + 1) = n$	,! 14 ( $\Rightarrow$ I: 8,13)	i
$n = 0 \vee \exists m (m + 1) = n$	,! 15 ( $\vee$ E: 4,7,14)	i
$\omega[n] \Rightarrow n = 0 \vee \exists m (m + 1) = n$	,! 16 ( $\Rightarrow$ I: 2,15)	i
$( \omega[n] \Rightarrow n = 0 \vee \exists m (m + 1) = n )$	,! 17 (( )I: 16)	i
$\forall n ( \omega[n] \Rightarrow n = 0 \vee \exists m (m + 1) = n )$	! 18 ( $\forall$ I: 1,17)	i

□

! 51. Addition of the successor is the same as the successor of the addition. i

$\vdash \forall n \forall m ( \omega[n] \ \& \ \omega[m] \Rightarrow (n + (m')) = ((n + m)') )$  i

**n,m** ,! 1 (Prem) i

$\omega[n] \ \& \ \omega[m]$  ,! 2 (Prem) i

$\omega[n] \ \& \ \omega[m] \ \& \ \omega[1]$  ,! 3 (&I: IV9.2,2) i

$( \omega[n] \ \& \ \omega[m] \ \& \ \omega[1] \Rightarrow (n + (m + 1)) = ((n + m) + 1) )$   
,! 4 ( $\forall$ E: P14) i

$\omega[n] \ \& \ \omega[m] \ \& \ \omega[1] \Rightarrow (n + (m + 1)) = ((n + m) + 1)$   
,! 5 (()E: 4) i

$(n + (m + 1)) = ((n + m) + 1)$  ,! 6 ( $\Rightarrow$ E: 3,5) i

$\omega[m]$  ,! 7 (&E: 2) i

$( \omega[m] \Rightarrow (m') = (m + 1) )$  ,! 8 ( $\forall$ E: P40) i

$\omega[m] \Rightarrow (m') = (m + 1)$  ,! 9 (()E: 8) i

$(m') = (m + 1)$  ,! 10 ( $\Rightarrow$ E: 7,9) i

$(n + (m')) = ((n + m) + 1)$  ,! 11 (=E: 6,10) i

$( \omega[n] \ \& \ \omega[m] \Rightarrow \omega[(n + m)] )$  ,! 12 ( $\forall$ E: C1.8) i

$\omega[n] \ \& \ \omega[m] \Rightarrow \omega[(n + m)]$  ,! 13 (()E: 12) i

$\omega[(n + m)]$  ,! 14 ( $\Rightarrow$ E: 2,13) i

$( \omega[(n + m)] \Rightarrow ((n + m) + 1) = ((n + m)') )$   
,! 15 ( $\forall$ E: P42;  
 $(n + m)$ : C1.7,2) i

$\omega[(n + m)] \Rightarrow ((n + m) + 1) = ((n + m)')$   
,! 16 (()E: 15) i

$((n + m) + 1) = ((n + m)')$  ,! 17 ( $\Rightarrow$ E: 14,16) i

$(n + (m')) = ((n + m)')$  ,! 18 (=E: 11,17) i

$\omega[n] \ \& \ \omega[m] \Rightarrow (n + (m')) = ((n + m)')$  ,! 19 ( $\Rightarrow$ I: 2,18) i

$( \omega[n] \ \& \ \omega[m] \Rightarrow (n + (m')) = ((n + m)') )$   
,! 20 (()I: 19) i

$\forall n \forall m ( \omega[n] \ \& \ \omega[m] \Rightarrow (n + (m')) = ((n + m)') )$   
! 21 ( $\forall$ I: 1,20) i

□

! 52. P52 is a corollary to P51. i

⊢  $\forall n \forall m \forall k ( (n + (m')) = k \Rightarrow ((n + m)') = k )$  i

**n, m, k** ,! 1 (Prem) i

$(n + (m')) = k$  ,! 2 (Prem) i

$\omega[n] \ \& \ \omega[(m')]$  ,! 3 (TE: C1.7,2) i

$\omega[n]$  ,! 4 (&E: 3) i

$\omega[(m')]$  ,! 5 (&E: 3) i

$\omega[m]$  ,! 6 (DP: IV8.12,5) i

$\omega[n] \ \& \ \omega[m]$  ,! 7 (&I: 4,6) i

$( \omega[n] \ \& \ \omega[m] \Rightarrow (n + (m')) = ((n + m)') )$   
,! 8 ( $\forall$ E: P51) i

$\omega[n] \ \& \ \omega[m] \Rightarrow (n + (m')) = ((n + m)')$   
,! 9 (( $\Rightarrow$ )E: 8) i

$(n + (m')) = ((n + m)')$  ,! 10 ( $\Rightarrow$ E: 7,9) i

$((n + m)') = k$  ,! 11 (=E: 2,10) i

$(n + (m')) = k \Rightarrow ((n + m)') = k$  ,! 12 ( $\Rightarrow$ I: 2,11) i

$( (n + (m')) = k \Rightarrow ((n + m)') = k )$  ,! 13 (( $\Rightarrow$ )I: 12) i

$\forall n \forall m \forall k ( (n + (m')) = k \Rightarrow ((n + m)') = k )$   
! 14 ( $\forall$ I: 1,13) i

□

! P53 through P55 show that only 0 plus 0 sums to 0. i

! 53. i

⊢  $\forall n \forall m ( (n + m) = 0 \Rightarrow m = 0 )$  i

**n, m** ,! 1 (Prem) i

$(n + m) = 0$  ,! 2 (Prem) i

$\neg m = 0$  ,! 3 (Prem) i

$\omega[n] \ \& \ \omega[m]$  ,! 4 (TE: C1.7,2) i

$\omega[n]$  ,! 5 (&E: 4) i

$\omega[m]$  ,! 6 (&E: 4) i

$\omega[\mathbf{m}] \ \& \ \neg \ \mathbf{m} = 0$	,! 7 (&I: 3,6)	i
$( \ \omega[\mathbf{m}] \ \& \ \neg \ \mathbf{m} = 0 \Rightarrow \exists \mathbf{m} \ ( \mathbf{m}' ) = \mathbf{m} )$	,! 8 ( $\forall$ E: IV8.34)	i
$\omega[\mathbf{m}] \ \& \ \neg \ \mathbf{m} = 0 \Rightarrow \exists \mathbf{m} \ ( \mathbf{m}' ) = \mathbf{m}$	,! 9 ( $( )$ E: 8)	i
$\exists \mathbf{m} \ ( \mathbf{m}' ) = \mathbf{m}$	,! 10 ( $\Rightarrow$ E: 7,9)	i
$( \mathbf{k}' ) = \mathbf{m}$	,! 11 ( $\exists$ E: 10)	i
$( \mathbf{n} + ( \mathbf{k}' ) ) = 0$	,! 12 ( $=$ E: 2,11)	i
$( ( \mathbf{n} + ( \mathbf{k}' ) ) = 0 \Rightarrow ( ( \mathbf{n} + \mathbf{k} )' ) = 0 )$	,! 13 ( $\forall$ E: P52)	i
$( \mathbf{n} + ( \mathbf{k}' ) ) = 0 \Rightarrow ( ( \mathbf{n} + \mathbf{k} )' ) = 0$	,! 14 ( $( )$ E: 13)	i
$( ( \mathbf{n} + \mathbf{k} )' ) = 0$	,! 15 ( $\Rightarrow$ E: 12,14)	i
$\omega[\mathbf{k}]$	,! 16 ( $\mathbb{D}$ P: IV8.12,11)	i
$\omega[\mathbf{n}] \ \& \ \omega[\mathbf{k}]$	,! 17 (&I: 5,16)	i
$\exists \mathbf{n} \ ( \mathbf{n}' ) = 0$	,! 18 ( $\exists$ I: C1.7,15,17)	i
$\mathfrak{F}$	,! 19 ( $\mathfrak{F}$ I: IV8.33,18)	i
$\neg \ \mathbf{m} = 0 \Rightarrow \mathfrak{F}$	,! 20 ( $\Rightarrow$ I: 3,19)	i
$\neg \neg \ \mathbf{m} = 0$	,! 21 ( $\neg$ I: 20)	i
$\mathbf{m} = 0$	,! 22 ( $\neg$ E: 21)	i
$( \mathbf{n} + \mathbf{m} ) = 0 \Rightarrow \mathbf{m} = 0$	,! 23 ( $\Rightarrow$ I: 2,22)	i
$( ( \mathbf{n} + \mathbf{m} ) = 0 \Rightarrow \mathbf{m} = 0 )$	,! 24 ( $( )$ I: 23)	i
$\forall \mathbf{n} \forall \mathbf{m} \ ( ( \mathbf{n} + \mathbf{m} ) = 0 \Rightarrow \mathbf{m} = 0 )$	! 25 ( $\forall$ I: 1,24)	i
$\square$		
<b>! 54.</b>		i
$\vdash \forall \mathbf{n} \forall \mathbf{m} \ ( ( \mathbf{n} + \mathbf{m} ) = 0 \Rightarrow \mathbf{n} = 0 )$		i
$\mathbf{n}, \mathbf{m}$	,! 1 (Prem)	i
$( \mathbf{n} + \mathbf{m} ) = 0$	,! 2 (Prem)	i
$( ( \mathbf{n} + \mathbf{m} ) = 0 \Rightarrow ( \mathbf{m} + \mathbf{n} ) = 0 )$	,! 3 ( $\forall$ E: P6)	i
$( \mathbf{n} + \mathbf{m} ) = 0 \Rightarrow ( \mathbf{m} + \mathbf{n} ) = 0$	,! 4 ( $( )$ E: 3)	i

$(\mathbf{m} + \mathbf{n}) = 0$	,! 5 ( $\Rightarrow$ E: 2,4)	i
$( (\mathbf{m} + \mathbf{n}) = 0 \Rightarrow \mathbf{n} = 0 )$	,! 6 ( $\forall$ E: P53)	i
$(\mathbf{m} + \mathbf{n}) = 0 \Rightarrow \mathbf{n} = 0$	,! 7 ( $($ )E: 6)	i
$\mathbf{n} = 0$	,! 8 ( $\Rightarrow$ E: 5,7)	i
$(\mathbf{n} + \mathbf{m}) = 0 \Rightarrow \mathbf{n} = 0$	,! 9 ( $\Rightarrow$ I: 2,8)	i
$( (\mathbf{n} + \mathbf{m}) = 0 \Rightarrow \mathbf{n} = 0 )$	,! 10 ( $($ )I: 9)	i
$\forall \mathbf{n} \forall \mathbf{m} ( (\mathbf{n} + \mathbf{m}) = 0 \Rightarrow \mathbf{n} = 0 )$	! 11 ( $\forall$ I: 1,10)	i

□

! 55.

$\vdash \forall \mathbf{n} \forall \mathbf{m} ( (\mathbf{n} + \mathbf{m}) = 0 \Rightarrow \mathbf{n} = 0 \ \& \ \mathbf{m} = 0 )$		i
$\mathbf{n}, \mathbf{m}$	,! 1 (Prem)	i
$(\mathbf{n} + \mathbf{m}) = 0$	,! 2 (Prem)	i
$( (\mathbf{n} + \mathbf{m}) = 0 \Rightarrow \mathbf{n} = 0 )$	,! 3 ( $\forall$ E: P54)	i
$(\mathbf{n} + \mathbf{m}) = 0 \Rightarrow \mathbf{n} = 0$	,! 4 ( $($ )E: 3)	i
$\mathbf{n} = 0$	,! 5 ( $\Rightarrow$ E: 2,4)	i
$( (\mathbf{n} + \mathbf{m}) = 0 \Rightarrow \mathbf{m} = 0 )$	,! 6 ( $\forall$ E: P53)	i
$(\mathbf{n} + \mathbf{m}) = 0 \Rightarrow \mathbf{m} = 0$	,! 7 ( $($ )E: 6)	i
$\mathbf{m} = 0$	,! 8 ( $\Rightarrow$ E: 2,7)	i
$\mathbf{n} = 0 \ \& \ \mathbf{m} = 0$	,! 9 ( $\Rightarrow$ E: 5,8)	i
$(\mathbf{n} + \mathbf{m}) = 0 \Rightarrow \mathbf{n} = 0 \ \& \ \mathbf{m} = 0$	,! 10 ( $\Rightarrow$ I: 2,9)	i
$( (\mathbf{n} + \mathbf{m}) = 0 \Rightarrow \mathbf{n} = 0 \ \& \ \mathbf{m} = 0 )$	,! 11 ( $($ )I: 10)	i
$\forall \mathbf{n} \forall \mathbf{m} ( (\mathbf{n} + \mathbf{m}) = 0 \Rightarrow \mathbf{n} = 0 \ \& \ \mathbf{m} = 0 )$	! 12 ( $\forall$ I: 1,11)	i

□

! P56 and P57 express that 0 does not succeed any number. i

! 56.

$\vdash \forall \mathbf{n} \forall \mathbf{m} ( \mathbf{m} = (\mathbf{n} + 1) \Rightarrow \neg \mathbf{m} = 0 )$		i
$\mathbf{n}, \mathbf{m}$	,! 1 (Prem)	i
$\mathbf{m} = (\mathbf{n} + 1)$	,! 2 (Prem)	i
$\mathbf{m} = 0$	,! 3 (Prem)	i

$(n + 1) = 0$	,! 4 (=E: 2,3)	i
$( (n + 1) = 0 \Rightarrow 1 = 0 )$	,! 5 ( $\forall$ E: P53)	i
$(n + 1) = 0 \Rightarrow 1 = 0$	,! 6 (()E: 5)	i
$1 = 0$	,! 7 ( $\Rightarrow$ E: 4,6)	i
$\mathfrak{F}$	,! 8 ( $\mathfrak{F}$ I: IV9.6,7)	i
$m = 0 \Rightarrow \mathfrak{F}$	,! 9 ( $\Rightarrow$ I: 3,8)	i
$\neg m = 0$	,! 10 ( $\neg$ I: 9)	i
$m = (n + 1) \Rightarrow \neg m = 0$	,! 11 ( $\Rightarrow$ I: 2,10)	i
$( m = (n + 1) \Rightarrow \neg m = 0 )$	,! 12 (()I: 11)	i
$\forall n \forall m ( m = (n + 1) \Rightarrow \neg m = 0 )$	! 13 ( $\forall$ I: 1,12)	i

□

! 57.

$\vdash \forall n \neg (n + 1) = 0$		i
<b>n</b>	,! 1 (Prem)	i
$(n + 1) = 0$	,! 2 (Prem)	i
$0 = 0$	,! 3 (=I)	i
$0 = (n + 1)$	,! 4 (=E: 2,3)	i
$( 0 = (n + 1) \Rightarrow \neg 0 = 0 )$	,! 5 ( $\forall$ E: P56)	i
$0 = (n + 1) \Rightarrow \neg 0 = 0$	,! 6 (()E: 5)	i
$\neg 0 = 0$	,! 7 ( $\Rightarrow$ E: 4,6)	i
$\mathfrak{F}$	,! 8 ( $\mathfrak{F}$ I: 3,7)	i
$(n + 1) = 0 \Rightarrow \mathfrak{F}$	,! 9 ( $\Rightarrow$ I: 2,8)	i
$\neg (n + 1) = 0$	,! 10 ( $\neg$ I: 9)	i
$\forall n \neg (n + 1) = 0$	! 11 ( $\forall$ I: 1,10)	i

□

! 58.

$\vdash \forall n \forall P ( \mathfrak{N}[(n + 1), P] \Rightarrow \exists x P[x] )$		i
<b>n, P</b>	,! 1 (Prem)	i
$\mathfrak{N}[(n + 1), P]$	,! 2 (Prem)	i

$\omega[n] \ \& \ \omega[1]$	,! 3 (TE: C1.7,2)	i
$\neg (n + 1) = 0$	,! 4 ( $\forall$ E: P57)	i
$\mathcal{N}[(n + 1), P] \ \& \ \neg (n + 1) = 0$	,! 5 (&I: 2,4)	i
$( \mathcal{N}[(n + 1), P] \ \& \ \neg (n + 1) = 0 \Rightarrow \exists x P[x] )$	,! 6 ( $\forall$ E: IV3.17; (n + 1): C1.7,2)	i
$\mathcal{N}[(n + 1), P] \ \& \ \neg (n + 1) = 0 \Rightarrow \exists x P[x]$	,! 7 ((E: 6)	i
$\exists x P[x]$	,! 8 ( $\Rightarrow$ E: 5,7)	i
$\mathcal{N}[(n + 1), P] \Rightarrow \exists x P[x]$	,! 9 ( $\Rightarrow$ I: 2,8)	i
$( \mathcal{N}[(n + 1), P] \Rightarrow \exists x P[x] )$	,! 10 ((I: 9)	i
$\forall n \forall P ( \mathcal{N}[(n + 1), P] \Rightarrow \exists x P[x] )$	! 11 ( $\forall$ E: 1,10)	i
$\square$		

! 59. The Law of Additive Cancellation for 1. i

$\vdash \forall n \forall m ( (n + 1) = (m + 1) \Rightarrow n = m )$		i
<b>n, m</b>	,! 1 (Prem)	i
$(n + 1) = (m + 1)$	,! 2 (Prem)	i
$\omega[n] \ \& \ \omega[1]$	,! 3 (TE: C1.7,2)	i
$\omega[n]$	,! 4 (&E: 3)	i
$( \omega[n] \Rightarrow (n') = (n + 1) )$	,! 5 ( $\forall$ E: P40)	i
$\omega[n] \Rightarrow (n') = (n + 1)$	,! 6 ((E: 5)	i
$(n') = (n + 1)$	,! 7 ( $\Rightarrow$ E: 4,6)	i
$(n') = (m + 1)$	,! 8 (=E: 2,7)	i
$\omega[m] \ \& \ \omega[1]$	,! 9 (TE: C1.7,8)	i
$\omega[m]$	,! 10 (&E: 9)	i
$( \omega[m] \Rightarrow (m') = (m + 1) )$	,! 11 ( $\forall$ E: P40)	i
$\omega[m] \Rightarrow (m') = (m + 1)$	,! 12 ((E: 11)	i
$(m') = (m + 1)$	,! 13 ( $\Rightarrow$ E: 10,12)	i
$(n') = (m')$	,! 14 (=E: 8,13)	i

$( (n') = (m') \Rightarrow n = m )$	,! 15 ( $\forall E$ : IV8.31)	i
$(n') = (m') \Rightarrow n = m$	,! 16 ( $(())E$ : 15)	i
$n = m$	,! 17 ( $\Rightarrow E$ : 14,16)	i
$(n + 1) = (m + 1) \Rightarrow n = m$	,! 18 ( $\Rightarrow I$ : 2,17)	i
$( (n + 1) = (m + 1) \Rightarrow n = m )$	,! 19 ( $(())I$ : 18)	i
$\forall n \forall m ( (n + 1) = (m + 1) \Rightarrow n = m )$	! 20 ( $\forall I$ : 19)	i

□

! P60 and P63 are the two Laws of Additive Cancellation. P61 and P62 are commutative variations. i

**! 60. Law of Right Additive Cancellation.** i

$\vdash \forall j \forall k \forall n ( (j + n) = (k + n) \Rightarrow j = k )$  i

! Fix  $j$  and  $k$  as  $\mathbf{j}$  and  $\mathbf{k}$ .

We first prove by induction that

$\forall n ( \omega[n] \Rightarrow ((\mathbf{j} + n) = (\mathbf{k} + n) \Rightarrow \mathbf{j} = \mathbf{k}) )$ .

To do this, it must be shown that

$((\mathbf{j} + 0) = (\mathbf{k} + 0) \Rightarrow \mathbf{j} = \mathbf{k})$

and

$\forall n \forall m ( \omega[n] \ \& \ \sigma[n,m] \ \& \ ((\mathbf{j} + n) = (\mathbf{k} + n) \Rightarrow \mathbf{j} = \mathbf{k})$		
$\Rightarrow ((\mathbf{j} + m) = (\mathbf{k} + m) \Rightarrow \mathbf{j} = \mathbf{k})$		i

$\mathbf{j}, \mathbf{k}$	,! 1 (Prem)	i
--------------------------	-------------	---

$(\mathbf{j} + 0) = (\mathbf{k} + 0)$	,! 2 (Prem)	i
---------------------------------------	-------------	---

$\omega[\mathbf{k}] \ \& \ \omega[0]$	,! 3 ( $\mathbb{T}E$ : C1.7,2)	i
---------------------------------------	--------------------------------	---

$( (\mathbf{j} + 0) = (\mathbf{k} + 0) \Rightarrow \mathbf{j} = (\mathbf{k} + 0) )$	,! 4 ( $\forall E$ : P34)	i
---	---------------------------	---

$(\mathbf{j} + 0) = (\mathbf{k} + 0) \Rightarrow \mathbf{j} = (\mathbf{k} + 0)$	,! 5 ( $(())E$ : 4)	i
---	---------------------	---

$\mathbf{j} = (\mathbf{k} + 0)$	,! 6 ( $\Rightarrow E$ : 2,5)	i
---------------------------------	-------------------------------	---

$( \mathbf{j} = (\mathbf{k} + 0) \Rightarrow \mathbf{j} = \mathbf{k} )$	,! 7 ( $\forall E$ : P35)	i
---	---------------------------	---

$\mathbf{j} = (\mathbf{k} + 0) \Rightarrow \mathbf{j} = \mathbf{k}$	,! 8 ( $(())E$ : 7)	i
---	---------------------	---

$\mathbf{j} = \mathbf{k}$	,! 9 ( $\Rightarrow E$ : 6,8)	i
---------------------------	-------------------------------	---

$(\mathbf{j} + 0) = (\mathbf{k} + 0) \Rightarrow \mathbf{j} = \mathbf{k}$	,! 10 ( $\Rightarrow I$ : 2,9)	i
---	--------------------------------	---

$((j + 0) = (k + 0) \Rightarrow j = k)$	,! 11 ((I: 10)	i
$n, m$	,! 12 (Prem)	i
$\omega[n] \ \& \ \sigma[n, m] \ \& \ ((j + n) = (k + n) \Rightarrow j = k)$	,! 13 (Prem)	i
$\omega[n] \ \& \ \sigma[n, m]$	,! 14 (&E: 13)	i
$((j + n) = (k + n) \Rightarrow j = k)$	,! 15 (&E: 13)	i
$(j + m) = (k + m)$	,! 16 (Prem)	i
$(\omega[n] \ \& \ \sigma[n, m] \Rightarrow (n + 1) = m)$	,! 17 ( $\forall$ E: P47)	i
$\omega[n] \ \& \ \sigma[n, m] \Rightarrow (n + 1) = m$	,! 18 ((E: 17)	i
$(n + 1) = m$	,! 19 ( $\Rightarrow$ E: 14,18)	i
$(j + (n + 1)) = (k + (n + 1))$	,! 20 (=E: 16,19)	i
$\omega[k] \ \& \ \omega[(n + 1)]$	,! 21 ( $\mathbb{T}$ E: C1.7,20)	i
$( (j + (n + 1)) = (k + (n + 1))$ $\Rightarrow ((j + n) + 1) = (k + (n + 1)) )$	,! 22 ( $\forall$ E: P15; $(k + (n + 1))$ : C1.7,21)	i
$(j + (n + 1)) = (k + (n + 1))$ $\Rightarrow ((j + n) + 1) = (k + (n + 1))$	,! 23 ((E: 22)	i
$((j + n) + 1) = (k + (n + 1))$	,! 24 ( $\Rightarrow$ E: 20,23)	i
$\omega[(j + n)] \ \& \ \omega[1]$	,! 25 ( $\mathbb{T}$ E: C1.7,24)	i
$( ((j + n) + 1) = (k + (n + 1))$ $\Rightarrow ((j + n) + 1) = ((k + n) + 1) )$	,! 26 ( $\forall$ E: P17)	i
$((j + n) + 1) = (k + (n + 1))$ $\Rightarrow ((j + n) + 1) = ((k + n) + 1)$	,! 27 ((E: 26)	i
$((j + n) + 1) = ((k + n) + 1)$	,! 28 ( $\Rightarrow$ E: 24,27)	i
$\omega[(j + n)]$	,! 29 (&E: 25)	i
$\omega[j] \ \& \ \omega[n]$	,! 30 ( $\mathbb{T}$ E: C1.7,29)	i
$\omega[(k + n)] \ \& \ \omega[1]$	,! 31 ( $\mathbb{T}$ E: C1.7,28)	i
$\omega[(k + n)]$	,! 32 (&E: 31)	i
$\omega[k] \ \& \ \omega[n]$	,! 33 ( $\mathbb{T}$ E: C1.7,32)	i

$((j + n) + 1) = ((k + n) + 1) \Rightarrow (j + n) = (k + n)$	,! 34 ( $\forall E$ : P59; (j + n): C1.7,30; (k + n): C1.7,33) i
$((j + n) + 1) = ((k + n) + 1) \Rightarrow (j + n) = (k + n)$	,! 35 ( $()E$ : 34) i
$(j + n) = (k + n)$	,! 36 ( $\Rightarrow E$ : 28,35) i
$(j + n) = (k + n) \Rightarrow j = k$	,! 37 ( $()E$ : 15) i
$j = k$	,! 38 ( $\Rightarrow E$ : 36,37) i
$(j + m) = (k + m) \Rightarrow j = k$	,! 39 ( $\Rightarrow I$ : 16,38) i
$((j + m) = (k + m) \Rightarrow j = k)$	,! 40 ( $()I$ : 39) i
$\omega[n] \ \& \ \sigma[n,m] \ \& \ ((j + n) = (k + n) \Rightarrow j = k)$ $\Rightarrow ((j + m) = (k + m) \Rightarrow j = k)$	,! 41 ( $\Rightarrow I$ : 13,40) i
$(\omega[n] \ \& \ \sigma[n,m] \ \& \ ((j + n) = (k + n) \Rightarrow j = k))$ $\Rightarrow ((j + m) = (k + m) \Rightarrow j = k)$	,! 42 ( $()I$ : 41) i
$\forall n \forall m (\omega[n] \ \& \ \sigma[n,m] \ \& \ ((j + n) = (k + n) \Rightarrow j = k))$ $\Rightarrow ((j + m) = (k + m) \Rightarrow j = k)$	,! 43 ( $\forall I$ : 12,42) i
$\forall n (\omega[n] \Rightarrow ((j + n) = (k + n) \Rightarrow j = k))$	,! 44 (Induct: 11,43) i
! Now to the proof of the proposition proper. i	
$n$	,! 45 (Prem) i
$(j + n) = (k + n)$	,! 46 (Prem) i
$\omega[j] \ \& \ \omega[n]$	,! 47 ( $\mathbb{T}E$ : C1.7,46) i
$\omega[n]$	,! 48 ( $\&E$ : 47) i
$(\omega[n] \Rightarrow ((j + n) = (k + n) \Rightarrow j = k))$	,! 49 ( $\forall E$ : 44) i
$\omega[n] \Rightarrow ((j + n) = (k + n) \Rightarrow j = k)$	,! 50 ( $()E$ : 49) i
$((j + n) = (k + n) \Rightarrow j = k)$	,! 51 ( $\Rightarrow E$ : 48,50) i
$(j + n) = (k + n) \Rightarrow j = k$	,! 52 ( $()E$ : 51) i

$j = k$	,! 53 ( $\Rightarrow$ E: 46,52)	i
$(j + n) = (k + n) \Rightarrow j = k$	,! 54 ( $\Rightarrow$ I: 46,53)	i
$((j + n) = (k + n) \Rightarrow j = k)$	,! 55 ( $(())$ I: 54)	i
$\forall n ((j + n) = (k + n) \Rightarrow j = k)$	,! 56 ( $\forall$ I: 45,55)	i
$\forall j \forall k \forall n ((j + n) = (k + n) \Rightarrow j = k)$	! 57 ( $\forall$ I: 1,56)	i
$\square$		

! 61. i

$\vdash \forall j \forall k \forall n ((j + n) = (n + k) \Rightarrow j = k)$		i
$j, k, n$	,! 1 (Prem)	i
$(j + n) = (n + k)$	,! 2 (Prem)	i
$\omega[j] \ \& \ \omega[n]$	,! 3 ( $\mathbb{T}$ E: C1.7,2)	i
$((j + n) = (n + k) \Rightarrow (j + n) = (k + n))$	,! 4 ( $\forall$ E: P7; (j + n): C1.7,3)	i
$(j + n) = (n + k) \Rightarrow (j + n) = (k + n)$	,! 5 ( $(())$ E: 4)	i
$(j + n) = (k + n)$	,! 6 ( $\Rightarrow$ E: 2,5)	i
$((j + n) = (k + n) \Rightarrow j = k)$	,! 7 ( $\forall$ E: P60)	i
$(j + n) = (k + n) \Rightarrow j = k$	,! 8 ( $(())$ E: 7)	i
$j = k$	,! 9 ( $\Rightarrow$ E: 6,8)	i
$(j + n) = (n + k) \Rightarrow j = k$	,! 10 ( $\Rightarrow$ I: 2,9)	i
$((j + n) = (n + k) \Rightarrow j = k)$	,! 11 ( $(())$ I: 10)	i
$\forall j \forall k \forall n ((j + n) = (n + k) \Rightarrow j = k)$	! 12 ( $\forall$ I: 1,11)	i
$\square$		

! 62. i

$\vdash \forall j \forall k \forall n ((n + j) = (k + n) \Rightarrow j = k)$		i
$j, k, n$	,! 1 (Prem)	i
$(n + j) = (k + n)$	,! 2 (Prem)	i
$\omega[k] \ \& \ \omega[n]$	,! 3 ( $\mathbb{T}$ E: C1.7,2)	i
$((n + j) = (k + n) \Rightarrow (j + n) = (k + n))$		

	,! 4 ( $\forall E$ : P6; ( $\mathbf{k} + \mathbf{n}$ ): C1.7,3)	i
$(\mathbf{n} + \mathbf{j}) = (\mathbf{k} + \mathbf{n}) \Rightarrow (\mathbf{j} + \mathbf{n}) = (\mathbf{k} + \mathbf{n})$	,! 5 ( $(\ )E$ : 4)	i
$(\mathbf{j} + \mathbf{n}) = (\mathbf{k} + \mathbf{n})$	,! 6 ( $\Rightarrow E$ : 2,5)	i
$( (\mathbf{j} + \mathbf{n}) = (\mathbf{k} + \mathbf{n}) \Rightarrow \mathbf{j} = \mathbf{k} )$	,! 7 ( $\forall E$ : P60)	i
$(\mathbf{j} + \mathbf{n}) = (\mathbf{k} + \mathbf{n}) \Rightarrow \mathbf{j} = \mathbf{k}$	,! 8 ( $(\ )E$ : 7)	i
$\mathbf{j} = \mathbf{k}$	,! 9 ( $\Rightarrow E$ : 6,8)	i
$(\mathbf{n} + \mathbf{j}) = (\mathbf{k} + \mathbf{n}) \Rightarrow \mathbf{j} = \mathbf{k}$	,! 10 ( $\Rightarrow I$ : 2,9)	i
$( (\mathbf{n} + \mathbf{j}) = (\mathbf{k} + \mathbf{n}) \Rightarrow \mathbf{j} = \mathbf{k} )$	,! 11 ( $(\ )I$ : 10)	i
$\forall \mathbf{j} \forall \mathbf{k} \forall \mathbf{n} ( (\mathbf{n} + \mathbf{j}) = (\mathbf{k} + \mathbf{n}) \Rightarrow \mathbf{j} = \mathbf{k} )$	! 12 ( $\forall I$ : 1,11)	i
$\square$		

**! 63. Law of Left Additive Cancellation.**

$\vdash \forall \mathbf{j} \forall \mathbf{k} \forall \mathbf{n} ( (\mathbf{n} + \mathbf{j}) = (\mathbf{n} + \mathbf{k}) \Rightarrow \mathbf{j} = \mathbf{k} )$		i
$\mathbf{j}, \mathbf{k}, \mathbf{n}$	,! 1 (Prem)	i
$(\mathbf{n} + \mathbf{j}) = (\mathbf{n} + \mathbf{k})$	,! 2 (Prem)	i
$\omega[\mathbf{n}] \ \& \ \omega[\mathbf{k}]$	,! 3 ( $\mathbb{T}E$ : C1.7,2)	i
$( (\mathbf{n} + \mathbf{j}) = (\mathbf{n} + \mathbf{k}) \Rightarrow (\mathbf{j} + \mathbf{n}) = (\mathbf{n} + \mathbf{k}) )$	,! 4 ( $\forall E$ : P6; ( $\mathbf{n} + \mathbf{k}$ ): C1.7,3)	i
$(\mathbf{n} + \mathbf{j}) = (\mathbf{n} + \mathbf{k}) \Rightarrow (\mathbf{j} + \mathbf{n}) = (\mathbf{n} + \mathbf{k})$	,! 5 ( $(\ )E$ : 4)	i
$(\mathbf{j} + \mathbf{n}) = (\mathbf{n} + \mathbf{k})$	,! 6 ( $\Rightarrow E$ : 2,5)	i
$( (\mathbf{j} + \mathbf{n}) = (\mathbf{n} + \mathbf{k}) \Rightarrow \mathbf{j} = \mathbf{k} )$	,! 7 ( $\forall E$ : P61)	i
$(\mathbf{j} + \mathbf{n}) = (\mathbf{n} + \mathbf{k}) \Rightarrow \mathbf{j} = \mathbf{k}$	,! 8 ( $(\ )E$ : 7)	i
$\mathbf{j} = \mathbf{k}$	,! 9 ( $\Rightarrow E$ : 6,8)	i
$(\mathbf{n} + \mathbf{j}) = (\mathbf{n} + \mathbf{k}) \Rightarrow \mathbf{j} = \mathbf{k}$	,! 10 ( $\Rightarrow I$ : 2,9)	i
$( (\mathbf{n} + \mathbf{j}) = (\mathbf{n} + \mathbf{k}) \Rightarrow \mathbf{j} = \mathbf{k} )$	,! 11 ( $(\ )I$ : 10)	i
$\forall \mathbf{j} \forall \mathbf{k} \forall \mathbf{n} ( (\mathbf{n} + \mathbf{j}) = (\mathbf{n} + \mathbf{k}) \Rightarrow \mathbf{j} = \mathbf{k} )$	! 12 ( $\forall I$ : 1,11)	i
$\square$		

**! 64.** P64 shows that Induction may be used to begin at any

finite a (and not just 0). i

$\vdash \forall P \forall a ( \omega[a] \ \& \ P[a] \ \& \ \forall n ( P[(a+n)] \Rightarrow P[(a+(n+1))] ) )$   
 $\Rightarrow \forall n ( \omega[n] \Rightarrow P[(a+n)] ) )$  i

**P, a** ,! 1 (Prem) i

$\omega[a] \ \& \ P[a] \ \& \ \forall n ( P[(a+n)] \Rightarrow P[(a+(n+1))] )$   
,! 2 (Prem) i

$\omega[a]$  ,! 3 (&E: 2) i

$P[a]$  ,! 4 (&E: 2) i

$\forall n ( P[(a+n)] \Rightarrow P[(a+(n+1))] )$  ,! 5 (&E: 2) i

! We proceed by induction, taking  $\phi$  to be

$P[(a+n)]$

It must be shown that

$P[(a+0)]$

and

$\forall n \forall m ( \omega[n] \ \& \ \sigma[n,m] \ \& \ P[(a+n)] \Rightarrow P[(a+m)] )$  i

! To prove:  $P[(a+0)]$  i

$( \omega[a] \Rightarrow (a + 0) = a )$  ,! 6 ( $\forall$ E: P32) i

$\omega[a] \Rightarrow (a + 0) = a$  ,! 7 (()E: 6) i

$(a + 0) = a$  ,! 8 ( $\Rightarrow$ E: 3,7) i

$P[(a+0)]$  ,! 9 (=E: 4,8) i

! To prove:  $\forall n \forall m ( \omega[n] \ \& \ \sigma[n,m] \ \& \ P[(a+n)] \Rightarrow P[(a+m)] )$  i

**n, m** ,! 10 (Prem) i

$\omega[n] \ \& \ \sigma[n,m] \ \& \ P[(a+n)]$  ,! 11 (Prem) i

$\omega[n] \ \& \ \sigma[n,m]$  ,! 12 (&E: 11) i

$P[(a+n)]$  ,! 13 (&E: 11) i

$( \omega[n] \ \& \ \sigma[n,m] \Rightarrow (n + 1) = m )$  ,! 14 ( $\forall$ E: P47) i

$\omega[n] \ \& \ \sigma[n,m] \Rightarrow (n + 1) = m$  ,! 15 (()E: 14) i

$(n + 1) = m$  ,! 16 ( $\Rightarrow$ E: 12,15) i

$( P[(a+n)] \Rightarrow P[(a+(n+1))] )$  ,! 17 ( $\forall$ E: 5) i

$P[(a+n)] \Rightarrow P[(a+(n+1))]$  ,! 18 (()E: 17) i

$P[(a+(n+1))]$  ,! 19 ( $\Rightarrow$ E: 13,18) i

$P[(a+m)]$  ,! 20 (=E: 16,19) i

$\omega[n] \ \& \ \sigma[n,m] \ \& \ P[(a+n)] \Rightarrow P[(a+m)]$	,! 21 ( $\Rightarrow$ I: 11,20)	i
$( \ \omega[n] \ \& \ \sigma[n,m] \ \& \ P[(a+n)] \Rightarrow P[(a+m)] \ )$	,! 22 ( $(())$ I: 21)	i
$\forall n \forall m ( \ \omega[n] \ \& \ \sigma[n,m] \ \& \ P[(a+n)] \Rightarrow P[(a+m)] \ )$	,! 23 ( $\forall$ I: 10,22)	i
$\forall n ( \ \omega[n] \Rightarrow P[(a+n)] \ )$	,! 24 (Induct: 9,23)	i
$\omega[a] \ \& \ P[a] \ \& \ \forall n ( P[(a+n)] \Rightarrow P[(a+(n+1))] )$		
$\Rightarrow \forall n ( \ \omega[n] \Rightarrow P[(a+n)] \ )$	,! 25 ( $\Rightarrow$ I: 2,24)	i
$( \ \omega[a] \ \& \ P[a] \ \& \ \forall n ( P[(a+n)] \Rightarrow P[(a+(n+1))] )$		
$\Rightarrow \forall n ( \ \omega[n] \Rightarrow P[(a+n)] \ )$	,! 26 ( $(())$ I: 25)	i
$\forall P \forall a ( \ \omega[a] \ \& \ P[a] \ \& \ \forall n ( P[(a+n)] \Rightarrow P[(a+(n+1))] )$		
$\Rightarrow \forall n ( \ \omega[n] \Rightarrow P[(a+n)] \ )$	,! 27 ( $\forall$ I: 1,26)	i
$\square$		
<b>! 65.</b>		i
$\vdash (0 + 0) = 0$		i
$( \ \omega[0] \Rightarrow (0 + 0) = 0 \ )$	,! 1 ( $\forall$ E: P32)	i
$\omega[0] \Rightarrow (0 + 0) = 0$	,! 2 ( $(())$ E: 1)	i
$(0 + 0) = 0$	! 3 ( $\Rightarrow$ E: $\omega 0, 2$ )	i
$\square$		
<b>! 66.</b>		i
$\vdash 0 = (0 + 0)$		i
$0 = 0$	,! 1 ( $=$ I)	i
$0 = (0 + 0)$	,! 2 ( $=$ E: P65,1)	i
$\square$		
<b>! 67.</b>		i
$\vdash (0 + 1) = 1$		i
$( \ \omega[1] \Rightarrow (0 + 1) = 1 \ )$	,! 1 ( $\forall$ E: P33)	i
$\omega[1] \Rightarrow (0 + 1) = 1$	,! 2 ( $(())$ E: 1)	i

$(0 + 1) = 1$  ! 3 ( $\Rightarrow$ E: IV9.2,2) ;

□

! 68. i

$\vdash (0 + 2) = 2$  i

(  $\omega[2] \Rightarrow (0 + 2) = 2$  ) ,! 1 ( $\forall$ E: P33) ;

$\omega[2] \Rightarrow (0 + 2) = 2$  ,! 2 ( $(\ )$ E: 1) ;

$(0 + 2) = 2$  ! 3 ( $\Rightarrow$ E: IV9.11,2) ;

□

! 69. i

$\vdash (1 + 1) = 2$  i

(  $\omega[1] \Rightarrow (1 + 1) = (1')$  ) ,! 1 ( $\forall$ E: P42) ;

$\omega[1] \Rightarrow (1 + 1) = (1')$  ,! 2 ( $(\ )$ E: 1) ;

$(1 + 1) = (1')$  ,! 3 ( $\Rightarrow$ E: IV9.2,2) ;

$(1 + 1) = 2$  ! 4 ( $=$ E: IV9.10,3) ;

□

! 70. i

$\vdash (2 + 1) = 3$  i

(  $\omega[2] \Rightarrow (2 + 1) = (2')$  ) ,! 1 ( $\forall$ E: P42) ;

$\omega[2] \Rightarrow (2 + 1) = (2')$  ,! 2 ( $(\ )$ E: 1) ;

$(2 + 1) = (2')$  ,! 3 ( $\Rightarrow$ E: IV9.11,2) ;

$(2 + 1) = 3$  ! 4 ( $=$ E: IV10.1,3) ;

□

! 71. i

$\vdash (1 + 2) = 3$  i

(  $(2 + 1) = 3 \Rightarrow (1 + 2) = 3$  ) ,! 1 ( $\forall$ E: P6) ;

$(2 + 1) = 3 \Rightarrow (1 + 2) = 3$  ,! 2 ( $(\ )$ E: 1) ;

$(1 + 2) = 3$  ,! 3 ( $\Rightarrow$ E: P70,2) ;

□

! 72.

$$\vdash (3 + 1) = 4$$

$$(\omega[3] \Rightarrow (3 + 1) = (3'))$$

$$\omega[3] \Rightarrow (3 + 1) = (3')$$

$$(3 + 1) = (3')$$

$$(3 + 1) = 4$$

,! 1 ( $\forall E$ : P42)

,! 2 ( $(\ )E$ : 1)

,! 3 ( $\Rightarrow E$ : IV10.2,2)

! 4 ( $=E$ : IV10.11,3)

□

! 73.

$$\vdash (1 + 3) = 4$$

$$((3 + 1) = 4 \Rightarrow (1 + 3) = 4)$$

$$(3 + 1) = 4 \Rightarrow (1 + 3) = 4$$

$$(1 + 3) = 4$$

,! 1 ( $\forall E$ : P6)

,! 2 ( $(\ )E$ : 1)

,! 3 ( $\Rightarrow E$ : P72,2)

□

! 74.

$$\vdash (2 + 2) = 4$$

$$((2 + 1) + 1) = 4$$

$$(((2 + 1) + 1) = 4 \Rightarrow (2 + (1 + 1)) = 4)$$

$$((2 + 1) + 1) = 4 \Rightarrow (2 + (1 + 1)) = 4$$

$$(2 + (1 + 1)) = 4$$

$$(2 + 2) = 4$$

,! 1 ( $=E$ : P70,P72)

,! 2 ( $\forall E$ : P16)

,! 3 ( $(\ )E$ : 2)

,! 4 ( $\Rightarrow E$ : 1,3)

! 5 ( $=E$ : P69,4)

□