

**DYADIC ADDITION, SUBTRACTION,  
AND MULTIPLICATION;**

! So far addition, subtraction, and multiplication have appeared as a triadic (3-place) relationship and as a term, defined by the **T**-rule. Now we introduce dyadic or 2-place relationships. For instance, when **c** is a finite number,  $(\oplus \mathbf{c})$  will represent dyadic addition, the adding of **c**. So **x** will hold  $(\oplus \mathbf{c})$  to **y** if and only if  $(x+c) = y$ . The advantage of doing so is it allows us to use the considerable machinery developed in Section III for dyadic relationships.

Highlights of this chapter:

P7: Adding 0 is the identity on  $\omega$

P8: Adding 1 is succession on  $\omega$

P23-P24: Dyadic addition and subtraction are inverses

P38: Multiplying by 1 is the identity on  $\omega$

! 1.  $(\oplus \mathbf{c})$  represents dyadic addition, the adding of **c**.

$\mathbb{D} \oplus ; (\oplus \mathbf{c}) ; ; ((\oplus \diamond \mathbf{c}) \lceil \omega) \lfloor \omega)$

! 2.

$\vdash \forall c \forall x \forall y ( \omega[c] \ \& \ (\oplus \mathbf{c})[x,y] \Leftrightarrow (x+c) = y )$

**c, x, y** ,! 1 (Prem)

$( \omega[\mathbf{x}] \ \& \ \omega[\mathbf{c}] \ \& \ \omega[\mathbf{y}] \ \& \ \oplus[\mathbf{x}, \mathbf{c}, \mathbf{y}] \Leftrightarrow (\mathbf{x}+\mathbf{c}) = \mathbf{y} )$

,! 2 ( $\forall E$ : C1.15)

$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{c}] \ \& \ \omega[\mathbf{y}] \ \& \ \oplus[\mathbf{x}, \mathbf{c}, \mathbf{y}] \Leftrightarrow (\mathbf{x}+\mathbf{c}) = \mathbf{y}$

,! 3 ( $(\ )E$ : 2)

$( (\oplus \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}] \Leftrightarrow \oplus[\mathbf{x}, \mathbf{c}, \mathbf{y}] )$

,! 4 ( $\forall E$ : III17.2)

$(\oplus \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}] \Leftrightarrow \oplus[\mathbf{x}, \mathbf{c}, \mathbf{y}]$

,! 5 ( $(\ )E$ : 4)

$( ((\oplus \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\oplus \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{x}] )$

,! 6 ( $\forall E$ : III7.2)

$((\oplus \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\oplus \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{x}]$

,! 7 ( $(\ )E$ : 6)

$( (((\oplus \diamond \mathbf{c}) \lceil \omega) \lfloor \omega)[\mathbf{x}, \mathbf{y}] \Leftrightarrow ((\oplus \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{y}] )$

,! 8 ( $\forall E$  III7.50)

$( (((\oplus \diamond \mathbf{c}) \lceil \omega) \lfloor \omega)[\mathbf{x}, \mathbf{y}] \Leftrightarrow ((\oplus \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{y}]$

,! 9 ( $(\ )E$ : 8)

$(\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}] \Leftrightarrow ((\oplus \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{y}]$  ,! 10 ( $\mathbb{D}I$ : P1,9)

$\omega[\mathbf{c}] \ \& \ (\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}]$

,! 11 (Prem)

$\omega[\mathbf{c}]$

,! 12 ( $\&E$ : 11)

$(\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}]$	, ! 13 (&E: 11)	i
$(\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}] \Rightarrow ((\oplus \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{y}]$	, ! 14 ( $\Leftrightarrow$ E: 10)	i
$((\oplus \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{y}]$	, ! 15 ( $\Rightarrow$ E: 13,14)	i
$((\oplus \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}]$	, ! 16 (&E: 15)	i
$\omega[\mathbf{y}]$	, ! 17 (&E: 15)	i
$((\oplus \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}] \Rightarrow (\oplus \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{x}]$	, ! 18 ( $\Leftrightarrow$ E: 7)	i
$(\oplus \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{x}]$	, ! 19 ( $\Rightarrow$ E: 16,18)	i
$(\oplus \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}]$	, ! 20 (&E: 19)	i
$\omega[\mathbf{x}]$	, ! 21 (&E: 19)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{c}]$	, ! 22 (&I: 12,21)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{c}] \ \& \ \omega[\mathbf{y}]$	, ! 23 (&I: 17,22)	i
$(\oplus \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}] \Rightarrow \oplus[\mathbf{x}, \mathbf{c}, \mathbf{y}]$	, ! 24 ( $\Leftrightarrow$ E: 5)	i
$\oplus[\mathbf{x}, \mathbf{c}, \mathbf{y}]$	, ! 25 ( $\Rightarrow$ E: 20,24)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{c}] \ \& \ \omega[\mathbf{y}] \ \& \ \oplus[\mathbf{x}, \mathbf{c}, \mathbf{y}]$	, ! 26 (&I: 23,25)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{c}] \ \& \ \omega[\mathbf{y}] \ \& \ \oplus[\mathbf{x}, \mathbf{c}, \mathbf{y}] \Rightarrow (\mathbf{x} + \mathbf{c}) = \mathbf{y}$	, ! 27 ( $\Leftrightarrow$ E: 3)	i
$(\mathbf{x} + \mathbf{c}) = \mathbf{y}$	, ! 28 ( $\Rightarrow$ E: 26,27)	i
$\omega[\mathbf{c}] \ \& \ (\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbf{x} + \mathbf{c}) = \mathbf{y}$	, ! 29 ( $\Rightarrow$ I: 11,28)	i
$(\mathbf{x} + \mathbf{c}) = \mathbf{y}$	, ! 30 (Prem)	i
$(\mathbf{x} + \mathbf{c}) = \mathbf{y} \Rightarrow \omega[\mathbf{x}] \ \& \ \omega[\mathbf{c}] \ \& \ \omega[\mathbf{y}] \ \& \ \oplus[\mathbf{x}, \mathbf{c}, \mathbf{y}]$	, ! 31 ( $\Leftrightarrow$ E: 3)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{c}] \ \& \ \omega[\mathbf{y}] \ \& \ \oplus[\mathbf{x}, \mathbf{c}, \mathbf{y}]$	, ! 32 ( $\Rightarrow$ E: 30,31)	i
$\omega[\mathbf{x}]$	, ! 33 (&E: 32)	i
$\omega[\mathbf{c}]$	, ! 34 (&E: 32)	i
$\omega[\mathbf{y}]$	, ! 35 (&E: 32)	i
$\oplus[\mathbf{x}, \mathbf{c}, \mathbf{y}]$	, ! 36 (&E: 32)	i

$\oplus[\mathbf{x}, \mathbf{c}, \mathbf{y}] \Rightarrow (\oplus \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}]$  ,! 37 ( $\Leftrightarrow E$ : 5) i  
 $(\oplus \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}]$  ,! 38 ( $\Rightarrow E$ : 36,37) i  
 $(\oplus \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{x}]$  ,! 39 ( $\& I$ : 33,38) i  
 $(\oplus \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{x}] \Rightarrow ((\oplus \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}]$  ,! 40 ( $\Leftrightarrow E$ : 7) i  
 $((\oplus \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}]$  ,! 41 ( $\Rightarrow E$ : 39,40) i  
 $((\oplus \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{y}]$  ,! 42 ( $\& I$ : 35,41) i  
 $((\oplus \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{y}] \Rightarrow (\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}]$  ,! 43 ( $\Leftrightarrow E$ : 10) i  
 $(\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}]$  ,! 44 ( $\Rightarrow E$ : 42,43) i  
 $\omega[\mathbf{c}] \ \& \ (\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}]$  ,! 45 ( $\& I$ : 34,44) i  
 $(\mathbf{x} + \mathbf{c}) = \mathbf{y} \Rightarrow \omega[\mathbf{c}] \ \& \ (\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}]$  ,! 46 ( $\Rightarrow I$ : 30,45) i  
 $\omega[\mathbf{c}] \ \& \ (\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\mathbf{x} + \mathbf{c}) = \mathbf{y}$  ,! 47 ( $\Leftrightarrow I$ : 29,46) i  
 $(\omega[\mathbf{c}] \ \& \ (\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\mathbf{x} + \mathbf{c}) = \mathbf{y})$  ,! 48 ( $(\ ) I$ : 47) i  
 $\forall c \forall x \forall y (\omega[c] \ \& \ (\oplus c)[x, y] \Leftrightarrow (x+c) = y)$  ! 49 ( $\forall I$ : 1,48) i

□

! 3. i

$\vdash \forall c \forall x \forall y (\omega[c] \ \& \ (\oplus c)[x, y] \Rightarrow (x+c) = y)$  i  
 $\mathbf{c}, \mathbf{x}, \mathbf{y}$  ,! 1 (Prem) i  
 $(\omega[\mathbf{c}] \ \& \ (\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\mathbf{x} + \mathbf{c}) = \mathbf{y})$  ,! 2 ( $\forall E$ : P2) i  
 $\omega[\mathbf{c}] \ \& \ (\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\mathbf{x} + \mathbf{c}) = \mathbf{y}$  ,! 3 ( $(\ ) E$ : 2) i  
 $\omega[\mathbf{c}] \ \& \ (\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbf{x} + \mathbf{c}) = \mathbf{y}$  ,! 4 ( $\Leftrightarrow E$ : 3) i  
 $(\omega[\mathbf{c}] \ \& \ (\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbf{x} + \mathbf{c}) = \mathbf{y})$  ,! 5 ( $(\ ) I$ : 4) i  
 $\forall c \forall x \forall y (\omega[c] \ \& \ (\oplus c)[x, y] \Rightarrow (x+c) = y)$  ! 6 ( $\forall I$ : 1,5) i

□

! 4. i

$\vdash \forall c \forall x \forall y ((x+c) = y \Rightarrow (\oplus c)[x, y])$  i  
 $\mathbf{c}, \mathbf{x}, \mathbf{y}$  ,! 1 (Prem) i  
 $(\mathbf{x} + \mathbf{c}) = \mathbf{y}$  ,! 2 (Prem) i

$(\omega[\mathbf{c}] \ \& \ (\oplus\mathbf{c})[\mathbf{x},\mathbf{y}] \Leftrightarrow (\mathbf{x}+\mathbf{c}) = \mathbf{y})$	, ! 3 ( $\forall\text{E}$ : P2)	i
$\omega[\mathbf{c}] \ \& \ (\oplus\mathbf{c})[\mathbf{x},\mathbf{y}] \Leftrightarrow (\mathbf{x}+\mathbf{c}) = \mathbf{y}$	, ! 4 ( $(\ )\text{E}$ : 3)	i
$(\mathbf{x}+\mathbf{c}) = \mathbf{y} \Rightarrow \omega[\mathbf{c}] \ \& \ (\oplus\mathbf{c})[\mathbf{x},\mathbf{y}]$	, ! 5 ( $\Leftrightarrow\text{E}$ : 4)	i
$\omega[\mathbf{c}] \ \& \ (\oplus\mathbf{c})[\mathbf{x},\mathbf{y}]$	, ! 6 ( $\Rightarrow\text{E}$ : 2,5)	i
$(\oplus\mathbf{c})[\mathbf{x},\mathbf{y}]$	, ! 7 ( $\&\text{E}$ : 6)	i
$(\mathbf{x}+\mathbf{c}) = \mathbf{y} \Rightarrow (\oplus\mathbf{c})[\mathbf{x},\mathbf{y}]$	, ! 8 ( $\Rightarrow\text{I}$ : 2,7)	i
$( (\mathbf{x}+\mathbf{c}) = \mathbf{y} \Rightarrow (\oplus\mathbf{c})[\mathbf{x},\mathbf{y}] )$	, ! 9 ( $(\ )\text{I}$ : 8)	i
$\forall\mathbf{c}\forall\mathbf{x}\forall\mathbf{y} ( (\mathbf{x}+\mathbf{c}) = \mathbf{y} \Rightarrow (\oplus\mathbf{c})[\mathbf{x},\mathbf{y}] )$	! 10 ( $\forall\text{I}$ : 1,9)	i

□

! 5.

$\vdash \forall\mathbf{c}\forall\mathbf{x} ( \omega[\mathbf{x}] \ \& \ \omega[\mathbf{c}] \Rightarrow (\oplus\mathbf{c})[\mathbf{x},(\mathbf{x}+\mathbf{c})] )$		i
$\mathbf{c}, \mathbf{x}$	, ! 1 (Prem)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{c}]$	, ! 2 (Prem)	i
$(\mathbf{x}+\mathbf{c}) = (\mathbf{x}+\mathbf{c})$	, ! 3 ( $=\text{I}$ ; $(\mathbf{x}+\mathbf{c})$ : C1.7,2)	i
$( (\mathbf{x}+\mathbf{c}) = (\mathbf{x}+\mathbf{c}) \Rightarrow (\oplus\mathbf{c})[\mathbf{x},(\mathbf{x}+\mathbf{c})] )$	, ! 4 ( $\forall\text{E}$ : P4; $(\mathbf{x}+\mathbf{c})$ : C1.7,2)	i
$(\mathbf{x}+\mathbf{c}) = (\mathbf{x}+\mathbf{c}) \Rightarrow (\oplus\mathbf{c})[\mathbf{x},(\mathbf{x}+\mathbf{c})]$	, ! 5 ( $(\ )\text{E}$ : 4)	i
$(\oplus\mathbf{c})[\mathbf{x},(\mathbf{x}+\mathbf{c})]$	, ! 6 ( $\Rightarrow\text{E}$ : 3,5)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{c}] \Rightarrow (\oplus\mathbf{c})[\mathbf{x},(\mathbf{x}+\mathbf{c})]$	, ! 7 ( $\Rightarrow\text{I}$ : 2,6)	i
$( \omega[\mathbf{x}] \ \& \ \omega[\mathbf{c}] \Rightarrow (\oplus\mathbf{c})[\mathbf{x},(\mathbf{x}+\mathbf{c})] )$	, ! 8 ( $(\ )\text{I}$ : 7)	i
$\forall\mathbf{c}\forall\mathbf{x} ( \omega[\mathbf{x}] \ \& \ \omega[\mathbf{c}] \Rightarrow (\oplus\mathbf{c})[\mathbf{x},(\mathbf{x}+\mathbf{c})] )$	! 9 ( $\forall\text{I}$ : 1,8)	i

□

! 6.

$\vdash \forall\mathbf{c}\forall\mathbf{x}\forall\mathbf{y} ( \omega[\mathbf{c}] \ \& \ (\oplus\mathbf{c})[\mathbf{x},\mathbf{y}] \Rightarrow \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] )$		i
$\mathbf{c}, \mathbf{x}, \mathbf{y}$	, ! 1 (Prem)	i
$\omega[\mathbf{c}] \ \& \ (\oplus\mathbf{c})[\mathbf{x},\mathbf{y}]$	, ! 2 (Prem)	i
$( \omega[\mathbf{c}] \ \& \ (\oplus\mathbf{c})[\mathbf{x},\mathbf{y}] \Rightarrow (\mathbf{x}+\mathbf{c}) = \mathbf{y} )$	, ! 3 ( $\forall\text{E}$ : P3)	i
$\omega[\mathbf{c}] \ \& \ (\oplus\mathbf{c})[\mathbf{x},\mathbf{y}] \Rightarrow (\mathbf{x}+\mathbf{c}) = \mathbf{y}$	, ! 4 ( $(\ )\text{E}$ : 3)	i

$(\mathbf{x}+\mathbf{c}) = \mathbf{y}$	,! 5 ( $\Rightarrow$ E: 2,4)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{c}]$	,! 6 ( $\mathbb{T}$ E: C1.7,5)	i
$\omega[\mathbf{x}]$	,! 7 ( $\&$ E: 6)	i
$( \ \omega[\mathbf{x}] \ \& \ \omega[\mathbf{c}] \ \Rightarrow \ \omega[(\mathbf{x}+\mathbf{c})] \ )$	,! 8 ( $\forall$ E: C1.8)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{c}] \ \Rightarrow \ \omega[(\mathbf{x}+\mathbf{c})]$	,! 9 ( $()$ E: 8)	i
$\omega[(\mathbf{x}+\mathbf{c})]$	,! 10 ( $\Rightarrow$ E: 6,9)	i
$\omega[\mathbf{y}]$	,! 11 ( $=$ E: 5,10)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}]$	,! 12 ( $\&$ I: 7,11)	i
$\omega[\mathbf{c}] \ \& \ (\oplus\mathbf{c})[\mathbf{x},\mathbf{y}] \ \Rightarrow \ \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}]$	,! 13 ( $\Rightarrow$ I: 2,12)	i
$( \ \omega[\mathbf{c}] \ \& \ (\oplus\mathbf{c})[\mathbf{x},\mathbf{y}] \ \Rightarrow \ \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ )$	,! 14 ( $()$ E: 13)	i
$\forall\mathbf{c}\forall\mathbf{x}\forall\mathbf{y} \ ( \ \omega[\mathbf{c}] \ \& \ (\oplus\mathbf{c})[\mathbf{x},\mathbf{y}] \ \Rightarrow \ \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ )$	! 15 ( $\forall$ I: 1,14)	i

□

! 7. Adding 0 is the identity on  $\omega$ .

$\vdash (\oplus 0) \equiv (\mathbb{I}\omega)$		i
$\mathbf{x}, \mathbf{y}$	,! 1 (Prem)	i
$( \ \omega[0] \ \& \ (\oplus 0)[\mathbf{x},\mathbf{y}] \ \Leftrightarrow \ (\mathbf{x}+0) = \mathbf{y} \ )$	,! 2 ( $\forall$ E: P2)	i
$\omega[0] \ \& \ (\oplus 0)[\mathbf{x},\mathbf{y}] \ \Leftrightarrow \ (\mathbf{x}+0) = \mathbf{y}$	,! 3 ( $()$ E: 2)	i
$( \ \omega[\mathbf{x}] \ \Rightarrow \ (\mathbf{x}+0) = \mathbf{x} \ )$	,! 4 ( $\forall$ E: C2.32)	i
$\omega[\mathbf{x}] \ \Rightarrow \ (\mathbf{x}+0) = \mathbf{x}$	,! 5 ( $()$ E: 4)	i
$( \ (\mathbb{I}\omega)[\mathbf{x},\mathbf{y}] \ \Leftrightarrow \ \mathbf{x} = \mathbf{y} \ \& \ \omega[\mathbf{x}] \ )$	,! 6 ( $\forall$ E: III11.17)	i
$(\mathbb{I}\omega)[\mathbf{x},\mathbf{y}] \ \Leftrightarrow \ \mathbf{x} = \mathbf{y} \ \& \ \omega[\mathbf{x}]$	,! 7 ( $()$ E: 6)	i
$(\oplus 0)[\mathbf{x},\mathbf{y}]$	,! 8 (Prem)	i
$\omega[0] \ \& \ (\oplus 0)[\mathbf{x},\mathbf{y}]$	,! 9 ( $\&$ I: $\omega 0, 8$ )	i
$\omega[0] \ \& \ (\oplus 0)[\mathbf{x},\mathbf{y}] \ \Rightarrow \ (\mathbf{x}+0) = \mathbf{y}$	,! 10 ( $\Leftrightarrow$ E: 3)	i
$(\mathbf{x}+0) = \mathbf{y}$	,! 11 ( $\Rightarrow$ E: 9,10)	i
$\omega[\mathbf{x}] \ \& \ \omega[0]$	,! 12 ( $\mathbb{T}$ E: C1.7,11)	i

$\omega[\mathbf{x}]$	,! 13 (&E: 12)	i
$(\mathbf{x}+0) = \mathbf{x}$	,! 14 ( $\Rightarrow$ E: 5,13)	i
$\mathbf{x} = \mathbf{y}$	,! 15 (=E: 11,14)	i
$\mathbf{x} = \mathbf{y} \ \& \ \omega[\mathbf{x}]$	,! 16 (&I: 13,15)	i
$\mathbf{x} = \mathbf{y} \ \& \ \omega[\mathbf{x}] \Rightarrow (\mathbb{I}\omega)[\mathbf{x},\mathbf{y}]$	,! 17 ( $\Leftrightarrow$ E: 7)	i
$(\mathbb{I}\omega)[\mathbf{x},\mathbf{y}]$	,! 18 ( $\Rightarrow$ E: 16,17)	i
$(\oplus 0)[\mathbf{x},\mathbf{y}] \Rightarrow (\mathbb{I}\omega)[\mathbf{x},\mathbf{y}]$	,! 19 ( $\Rightarrow$ I: 8,18)	i
$(\mathbb{I}\omega)[\mathbf{x},\mathbf{y}]$	,! 20 (Prem)	i
$(\mathbb{I}\omega)[\mathbf{x},\mathbf{y}] \Rightarrow \mathbf{x} = \mathbf{y} \ \& \ \omega[\mathbf{x}]$	,! 21 ( $\Leftrightarrow$ E: 7)	i
$\mathbf{x} = \mathbf{y} \ \& \ \omega[\mathbf{x}]$	,! 22 ( $\Rightarrow$ E: 20,21)	i
$\mathbf{x} = \mathbf{y}$	,! 23 (&E: 22)	i
$\omega[\mathbf{x}]$	,! 24 (&E: 22)	i
$(\mathbf{x}+0) = \mathbf{x}$	,! 25 ( $\Rightarrow$ E: 5,24)	i
$(\mathbf{x}+0) = \mathbf{y}$	,! 26 (=E: 23,25)	i
$(\mathbf{x}+0) = \mathbf{y} \Rightarrow \omega[0] \ \& \ (\oplus 0)[\mathbf{x},\mathbf{y}]$	,! 27 ( $\Leftrightarrow$ E: 3)	i
$\omega[0] \ \& \ (\oplus 0)[\mathbf{x},\mathbf{y}]$	,! 28 ( $\Rightarrow$ E: 26,27)	i
$(\oplus 0)[\mathbf{x},\mathbf{y}]$	,! 29 (&E: 28)	i
$(\mathbb{I}\omega)[\mathbf{x},\mathbf{y}] \Rightarrow (\oplus 0)[\mathbf{x},\mathbf{y}]$	,! 30 ( $\Rightarrow$ I: 20,29)	i
$(\oplus 0)[\mathbf{x},\mathbf{y}] \Leftrightarrow (\mathbb{I}\omega)[\mathbf{x},\mathbf{y}]$	,! 31 ( $\Leftrightarrow$ I: 19,30)	i
$( (\oplus 0)[\mathbf{x},\mathbf{y}] \Leftrightarrow (\mathbb{I}\omega)[\mathbf{x},\mathbf{y}] )$	,! 32 (( )I: 31)	i
$\forall \mathbf{x} \forall \mathbf{y} ( (\oplus 0)[\mathbf{x},\mathbf{y}] \Leftrightarrow (\mathbb{I}\omega)[\mathbf{x},\mathbf{y}] )$	,! 33 ( $\forall$ I: 1,32)	i
$(\oplus 0) \equiv (\mathbb{I}\omega)$	! 34 ( $\S$ I: III1.5,33)	i

□

! 8. Adding 1 is succession on  $\omega$ .

$\vdash (\oplus 1) \equiv (\sigma \uparrow \omega)$		i
$\mathbf{x}, \mathbf{y}$	,! 1 (Prem)	i
$( \omega[1] \ \& \ (\oplus 1)[\mathbf{x},\mathbf{y}] \Leftrightarrow (\mathbf{x} + 1) = \mathbf{y} )$	,! 2 ( $\forall$ E: P2)	i

$\omega[1] \ \& \ (\oplus 1)[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\mathbf{x} + 1) = \mathbf{y}$	,! 3 ((E: 2)	i
$(\mathbf{x}') = \mathbf{y} \Leftrightarrow (\mathbf{x} + 1) = \mathbf{y}$	,! 4 ( $\forall$ E: C2.44)	i
$(\mathbf{x}') = \mathbf{y} \Leftrightarrow (\mathbf{x} + 1) = \mathbf{y}$	,! 5 ((E: 4)	i
$(\mathbf{x}') = \mathbf{y} \Leftrightarrow (\sigma \lceil \omega)[\mathbf{x}, \mathbf{y}]$	,! 6 ( $\forall$ E: IV8.20)	i
$(\mathbf{x}') = \mathbf{y} \Leftrightarrow (\sigma \lceil \omega)[\mathbf{x}, \mathbf{y}]$	,! 7 ((E: 6)	i
$(\oplus 1)[\mathbf{x}, \mathbf{y}]$	,! 8 (Prem)	i
$\omega[1] \ \& \ (\oplus 1)[\mathbf{x}, \mathbf{y}]$	,! 9 (&I: IV9.2,8)	i
$\omega[1] \ \& \ (\oplus 1)[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbf{x}+1) = \mathbf{y}$	,! 10 ( $\Leftrightarrow$ E: 3)	i
$(\mathbf{x}+1) = \mathbf{y}$	,! 11 ( $\Rightarrow$ E: 9,10)	i
$(\mathbf{x}+1) = \mathbf{y} \Rightarrow (\mathbf{x}') = \mathbf{y}$	,! 12 ( $\Leftrightarrow$ E: 5)	i
$(\mathbf{x}') = \mathbf{y}$	,! 13 ( $\Rightarrow$ E: 11,12)	i
$(\mathbf{x}') = \mathbf{y} \Rightarrow (\sigma \lceil \omega)[\mathbf{x}, \mathbf{y}]$	,! 14 ( $\Leftrightarrow$ E: 7)	i
$(\sigma \lceil \omega)[\mathbf{x}, \mathbf{y}]$	,! 15 ( $\Rightarrow$ E: 13,14)	i
$(\oplus 1)[\mathbf{x}, \mathbf{y}] \Rightarrow (\sigma \lceil \omega)[\mathbf{x}, \mathbf{y}]$	,! 16 ( $\Rightarrow$ I: 8,15)	i
$(\sigma \lceil \omega)[\mathbf{x}, \mathbf{y}]$	,! 17 (Prem)	i
$(\sigma \lceil \omega)[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbf{x}') = \mathbf{y}$	,! 18 ( $\Leftrightarrow$ E: 7)	i
$(\mathbf{x}') = \mathbf{y}$	,! 19 ( $\Rightarrow$ E: 17,18)	i
$(\mathbf{x}') = \mathbf{y} \Rightarrow (\mathbf{x} + 1) = \mathbf{y}$	,! 20 ( $\Leftrightarrow$ E: 5)	i
$(\mathbf{x} + 1) = \mathbf{y}$	,! 21 ( $\Rightarrow$ E: 19,20)	i
$(\mathbf{x} + 1) = \mathbf{y} \Rightarrow \omega[1] \ \& \ (\oplus 1)[\mathbf{x}, \mathbf{y}]$	,! 22 ( $\Leftrightarrow$ E: 3)	i
$\omega[1] \ \& \ (\oplus 1)[\mathbf{x}, \mathbf{y}]$	,! 23 ( $\Rightarrow$ E: 21,22)	i
$(\oplus 1)[\mathbf{x}, \mathbf{y}]$	,! 24 (&E: 23)	i
$(\sigma \lceil \omega)[\mathbf{x}, \mathbf{y}] \Rightarrow (\oplus 1)[\mathbf{x}, \mathbf{y}]$	,! 25 ( $\Rightarrow$ I: 17,24)	i
$(\oplus 1)[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\sigma \lceil \omega)[\mathbf{x}, \mathbf{y}]$	,! 26 ( $\Leftrightarrow$ I: 16,25)	i
$(\oplus 1)[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\sigma \lceil \omega)[\mathbf{x}, \mathbf{y}]$	,! 27 ((I: 26)	i
$\forall \mathbf{x} \forall \mathbf{y} \ ( \oplus 1)[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\sigma \lceil \omega)[\mathbf{x}, \mathbf{y}]$	,! 28 ( $\forall$ I: 1,27)	i
$(\oplus 1) \equiv (\sigma \lceil \omega)$	! 29 ( $\S$ I: III1.5,28)	i

□

! 9. i

⊢  $\forall c ( \omega[c] \Rightarrow ((\oplus c)^D) \equiv \omega )$  i

$c$  ,! 1 (Prem) i

$\omega[c]$  ,! 2 (Prem) i

$( \forall x(\exists y (\oplus c)[x,y] \Leftrightarrow \omega[x]) \Rightarrow ((\oplus c)^D) \equiv \omega )$   
,! 3 ( $\forall E$ : III5.13) i

$\forall x(\exists y (\oplus c)[x,y] \Leftrightarrow \omega[x]) \Rightarrow ((\oplus c)^D) \equiv \omega$   
,! 4 ( $()E$ : 3) i

$x$  ,! 5 (Prem) i

$\exists y (\oplus c)[x,y]$  ,! 6 (Prem) i

$(\oplus c)[x,y]$  ,! 7 ( $\exists E$ : 6) i

$\omega[c] \ \& \ (\oplus c)[x,y]$  ,! 8 ( $\&I$ : 2,7) i

$( \omega[c] \ \& \ (\oplus c)[x,y] \Rightarrow (\mathbf{x}+c) = \mathbf{y} )$  ,! 9 ( $\forall E$ : P3) i

$\omega[c] \ \& \ (\oplus c)[x,y] \Rightarrow (\mathbf{x}+c) = \mathbf{y}$  ,! 10 ( $()E$ : 9) i

$(\mathbf{x}+c) = \mathbf{y}$  ,! 11 ( $\Rightarrow E$ : 8,10) i

$\omega[\mathbf{x}] \ \& \ \omega[c]$  ,! 12 ( $\mathbb{T}E$ : C1.7,11) i

$\omega[\mathbf{x}]$  ,! 13 ( $\&E$ : 12) i

$\exists y (\oplus c)[x,y] \Rightarrow \omega[\mathbf{x}]$  ,! 14 ( $\Rightarrow I$ : 6,13) i

$\omega[\mathbf{x}]$  ,! 15 (Prem) i

$\omega[\mathbf{x}] \ \& \ \omega[c]$  ,! 16 ( $\&I$ : 2,15) i

$( \omega[\mathbf{x}] \ \& \ \omega[c] \Rightarrow (\oplus c)[\mathbf{x},(\mathbf{x}+c)] )$  ,! 17 ( $\forall E$ : P5) i

$\omega[\mathbf{x}] \ \& \ \omega[c] \Rightarrow (\oplus c)[\mathbf{x},(\mathbf{x}+c)]$  ,! 18 ( $()E$ : 17) i

$(\oplus c)[\mathbf{x},(\mathbf{x}+c)]$  ,! 19 ( $\Rightarrow E$ : 16,18) i

$\exists y (\oplus c)[x,y]$  ,! 20 ( $\exists I$ : 19;  
 $(\mathbf{x}+c)$ : C1.7,16) i

$\omega[\mathbf{x}] \Rightarrow \exists y (\oplus c)[x,y]$  ,! 21 ( $\Rightarrow I$ : 15,20) i

$\exists y (\oplus c)[x,y] \Leftrightarrow \omega[\mathbf{x}]$  ,! 22 ( $\Leftrightarrow I$ : 14,21) i

$(\exists y (\oplus c)[x,y] \Leftrightarrow \omega[\mathbf{x}])$  ,! 23 ( $()I$ : 22) i

$\forall x(\exists y (\oplus c)[x,y] \Leftrightarrow \omega[x])$	, ! 24 ( $\forall I$ : 5,23)	i
$((\oplus c)^D) \equiv \omega$	, ! 25 ( $\Rightarrow E$ : 4,24)	i
$\omega[c] \Rightarrow ((\oplus c)^D) \equiv \omega$	, ! 26 ( $\Rightarrow I$ : 2,25)	i
$(\omega[c] \Rightarrow ((\oplus c)^D) \equiv \omega)$	, ! 27 ( $((I$ : 26)	i
$\forall c (\omega[c] \Rightarrow ((\oplus c)^D) \equiv \omega)$	! 28 ( $\forall I$ : 1,27)	i
$\square$		
<b>! 10.</b>		i
$\vdash \forall c (\omega[c] \Rightarrow f(\oplus c))$		i
<b>c</b>	, ! 1 (Prem)	i
<b><math>\omega[c]</math></b>	, ! 2 (Prem)	i
<b><math>x, y, z</math></b>	, ! 3 (Prem)	i
<b><math>(\oplus c)[x,y] \ \&amp; \ (\oplus c)[x,z]</math></b>	, ! 4 (Prem)	i
<b><math>(\oplus c)[x,y]</math></b>	, ! 5 ( $\&E$ : 2)	i
<b><math>(\oplus c)[x,z]</math></b>	, ! 6 ( $\&E$ : 2)	i
<b><math>\omega[c] \ \&amp; \ (\oplus c)[x,y]</math></b>	, ! 7 ( $\&I$ : 2,5)	i
<b><math>(\omega[c] \ \&amp; \ (\oplus c)[x,y] \Rightarrow (x+c) = y)</math></b>	, ! 8 ( $\forall E$ : P3)	i
<b><math>\omega[c] \ \&amp; \ (\oplus c)[x,y] \Rightarrow (x+c) = y</math></b>	, ! 9 ( $((E$ : 8)	i
<b><math>(x+c) = y</math></b>	, ! 10 ( $\Rightarrow E$ : 7,9)	i
<b><math>\omega[c] \ \&amp; \ (\oplus c)[x,z]</math></b>	, ! 11 ( $\&I$ : 2,6)	i
<b><math>(\omega[c] \ \&amp; \ (\oplus c)[x,y] \Rightarrow (x+c) = z)</math></b>	, ! 12 ( $\forall E$ : P3)	i
<b><math>\omega[c] \ \&amp; \ (\oplus c)[x,y] \Rightarrow (x+c) = z</math></b>	, ! 13 ( $((E$ : 12)	i
<b><math>(x+c) = z</math></b>	, ! 14 ( $\Rightarrow E$ : 11,13)	i
<b><math>y = z</math></b>	, ! 15 ( $=E$ : 10,14)	i
<b><math>(\oplus c)[x,y] \ \&amp; \ (\oplus c)[x,z] \Rightarrow y = z</math></b>	, ! 16 ( $\Rightarrow I$ : 4,15)	i
<b><math>((\oplus c)[x,y] \ \&amp; \ (\oplus c)[x,z] \Rightarrow y = z)</math></b>	, ! 17 ( $((I$ : 16)	i
<b><math>\forall x \forall y \forall z ((\oplus c)[x,y] \ \&amp; \ (\oplus c)[x,z] \Rightarrow y = z)</math></b>	, ! 18 ( $\forall I$ : 3,17)	i
<b><math>f(\oplus c)</math></b>	, ! 19 ( $\$I$ : III8.1,18)	

			i
$\omega[c] \Rightarrow f(\oplus c)$	,!	20 ( $\Rightarrow I: 2,19$ )	i
$(\omega[c] \Rightarrow f(\oplus c))$	,!	21 ( $(\ ) I: 20$ )	i
$\forall c (\omega[c] \Rightarrow f(\oplus c))$	!	22 ( $\forall I: 1,21$ )	i
$\square$			
<b>! 11.</b>			i
$\vdash \forall c (\omega[c] \Rightarrow (\oplus c) \mathbf{F} \omega)$			i
<b>c</b>	,!	1 (Prem)	i
$\omega[c]$	,!	2 (Prem)	i
$(\omega[c] \Rightarrow ((\oplus c)^D) \equiv \omega)$	,!	3 ( $\forall E: P9$ )	i
$\omega[c] \Rightarrow ((\oplus c)^D) \equiv \omega$	,!	4 ( $(\ ) E: 3$ )	i
$((\oplus c)^D) \equiv \omega$	,!	5 ( $\Rightarrow E: 2,4$ )	i
$(\omega[c] \Rightarrow f(\oplus c))$	,!	6 ( $\forall E: P10$ )	i
$\omega[c] \Rightarrow f(\oplus c)$	,!	7 ( $(\ ) E: 6$ )	i
<b>f</b> ( $\oplus c$ )	,!	8 ( $\Rightarrow E: 2,7$ )	i
$((\oplus c)^D) \equiv \omega \ \& \ f(\oplus c)$	,!	9 ( $\& I: 5,8$ )	i
$(\oplus c) \mathbf{F} \omega$	,!	10 ( $\$ I: III8.10,9$ )	i
$\omega[c] \Rightarrow (\oplus c) \mathbf{F} \omega$	,!	11 ( $\Rightarrow I: 2,10$ )	i
$(\omega[c] \Rightarrow (\oplus c) \mathbf{F} \omega)$	,!	12 ( $(\ ) I: 11$ )	i
$\forall c (\omega[c] \Rightarrow (\oplus c) \mathbf{F} \omega)$	!	13 ( $\forall I: 1,12$ )	i
$\square$			
<b>! 12.</b>			i
$\vdash \forall c \forall x (\omega[x] \ \& \ \omega[c] \Rightarrow ((\oplus c)'x) = (x+c))$			i
<b>c, x</b>	,!	1 (Prem)	i
$\omega[x] \ \& \ \omega[c]$	,!	2 (Prem)	i
$\omega[c]$	,!	3 ( $\& E: 2$ )	i
$(\omega[c] \Rightarrow f(\oplus c))$	,!	4 ( $\forall E: P10$ )	i
$\omega[c] \Rightarrow f(\oplus c)$	,!	5 ( $(\ ) E: 4$ )	i

$f (\oplus c)$	, ! 6 ( $\Rightarrow E$ : 3,5)	i
$( \omega[x] \ \& \ \omega[c] \Rightarrow (\oplus c)[x, (x+c)] )$	, ! 7 ( $\forall E$ : P5)	i
$\omega[x] \ \& \ \omega[c] \Rightarrow (\oplus c)[x, (x+c)]$	, ! 8 ( $( )E$ : 7)	i
$(\oplus c)[x, (x+c)]$	, ! 9 ( $\Rightarrow E$ : 2,8)	i
$f (\oplus c) \ \& \ (\oplus c)[x, (x+c)]$	, ! 10 ( $\&I$ : 6,9)	i
$( f (\oplus c) \ \& \ (\oplus c)[x, (x+c)] \Rightarrow ((\oplus c)'x) = (x+c) )$	, ! 11 ( $\forall E$ : III8.22; ( $x+c$ ): C1.7,2)	i
$f (\oplus c) \ \& \ (\oplus c)[x, (x+c)] \Rightarrow ((\oplus c)'x) = (x+c)$	, ! 12 ( $( )E$ : 11)	i
$((\oplus c)'x) = (x+c)$	, ! 13 ( $\Rightarrow E$ : 10,12)	i
$\omega[x] \ \& \ \omega[c] \Rightarrow ((\oplus c)'x) = (x+c)$	, ! 14 ( $\Rightarrow I$ : 2,13)	i
$( \omega[x] \ \& \ \omega[c] \Rightarrow ((\oplus c)'x) = (x+c) )$	, ! 15 ( $( )I$ : 14)	i
$\forall c \forall x ( \omega[x] \ \& \ \omega[c] \Rightarrow ((\oplus c)'x) = (x+c) )$	! 16 ( $\forall I$ : 1,15)	i
$\square$		
! 13.		
$\vdash \forall c ( \omega[c] \Rightarrow ((\oplus c)^I) \equiv (c \ \infty) )$		
$c$	, ! 1 (Prem)	i
$\omega[c]$	, ! 2 (Prem)	i
$( \forall y (\exists x (\oplus c)[x, y] \Leftrightarrow (c \ \infty)[y]) \Rightarrow ((\oplus c)^I) \equiv (c \ \infty) )$	, ! 3 ( $\forall E$ : III6.14)	i
$\forall y (\exists x (\oplus c)[x, y] \Leftrightarrow (c \ \infty)[y]) \Rightarrow ((\oplus c)^I) \equiv (c \ \infty)$	, ! 4 ( $( )E$ : 3)	i
$y$	, ! 5 (Prem)	i
$( \omega[c] \ \& \ (\oplus c)[x, y] \Leftrightarrow (x+c) = y )$	, ! 6 ( $\forall E$ : P2)	i
$\omega[c] \ \& \ (\oplus c)[x, y] \Leftrightarrow (x+c) = y$	, ! 7 ( $( )E$ : 6)	i
$( \leq[c, y] \Leftrightarrow \exists k (k + c) = y )$	, ! 8 ( $\forall E$ : C3.18)	i
$\leq[c, y] \Leftrightarrow \exists k (k + c) = y$	, ! 9 ( $( )E$ : 8)	i
$( (c \ \infty)[y] \Leftrightarrow \leq[c, y] )$	, ! 10 ( $\forall E$ : C9.2)	i
$(c \ \infty)[y] \Leftrightarrow \leq[c, y]$	, ! 11 ( $( )E$ : 10)	i

$\exists x (\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}]$	,! 12 (Prem)	i
$(\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}]$	,! 13 ( $\exists E$ : 12)	i
$\omega[\mathbf{c}] \ \& \ (\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}]$	,! 14 ( $\&I$ : 2,13)	i
$\omega[\mathbf{c}] \ \& \ (\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbf{x} + \mathbf{c}) = \mathbf{y}$	,! 15 ( $\Leftrightarrow E$ : 7)	i
$(\mathbf{x} + \mathbf{c}) = \mathbf{y}$	,! 16 ( $\Rightarrow E$ : 14,15)	i
$\exists k (k + \mathbf{c}) = \mathbf{y}$	,! 17 ( $\exists I$ : 16)	i
$\exists k (k + \mathbf{c}) = \mathbf{y} \Rightarrow \leq[\mathbf{c}, \mathbf{y}]$	,! 18 ( $\Leftrightarrow E$ : 9)	i
$\leq[\mathbf{c}, \mathbf{y}]$	,! 19 ( $\Rightarrow E$ : 17,18)	i
$\leq[\mathbf{c}, \mathbf{y}] \Rightarrow (\mathbf{c} \ \infty)[\mathbf{y}]$	,! 20 ( $\Leftrightarrow E$ : 11)	i
$(\mathbf{c} \ \infty)[\mathbf{y}]$	,! 21 ( $\Rightarrow E$ : 19,20)	i
$\exists x (\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbf{c} \ \infty)[\mathbf{y}]$	,! 22 ( $\Rightarrow I$ : 12,21)	i
$(\mathbf{c} \ \infty)[\mathbf{y}]$	,! 23 (Prem)	i
$(\mathbf{c} \ \infty)[\mathbf{y}] \Rightarrow \leq[\mathbf{c}, \mathbf{y}]$	,! 24 ( $\Leftrightarrow E$ : 11)	i
$\leq[\mathbf{c}, \mathbf{y}]$	,! 25 ( $\Rightarrow E$ : 23,24)	i
$\leq[\mathbf{c}, \mathbf{y}] \Rightarrow \exists k (k + \mathbf{c}) = \mathbf{y}$	,! 26 ( $\Leftrightarrow E$ : 9)	i
$\exists k (k + \mathbf{c}) = \mathbf{y}$	,! 27 ( $\Rightarrow E$ : 25,26)	i
$(\mathbf{x} + \mathbf{c}) = \mathbf{y}$	,! 28 ( $\exists E$ : 27)	i
$(\mathbf{x} + \mathbf{c}) = \mathbf{y} \Rightarrow \omega[\mathbf{c}] \ \& \ (\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}]$	,! 29 ( $\Leftrightarrow E$ : 7)	i
$\omega[\mathbf{c}] \ \& \ (\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}]$	,! 30 ( $\Rightarrow E$ : 28,29)	i
$(\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}]$	,! 31 ( $\&E$ : 30)	i
$\exists x (\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}]$	,! 32 ( $\exists I$ : 31)	i
$(\mathbf{c} \ \infty)[\mathbf{y}] \Rightarrow \exists x (\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}]$	,! 33 ( $\Rightarrow I$ : 23,32)	i
$\exists x (\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\mathbf{c} \ \infty)[\mathbf{y}]$	,! 34 ( $\Leftrightarrow I$ : 22,33)	i
$(\exists x (\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\mathbf{c} \ \infty)[\mathbf{y}])$	,! 35 ( $(\ )I$ : 34)	i
$\forall \mathbf{y} (\exists x (\oplus \mathbf{c})[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\mathbf{c} \ \infty)[\mathbf{y}])$	,! 36 ( $\forall I$ : 5,35)	i
$((\oplus \mathbf{c})^I) \equiv (\mathbf{c} \ \infty)$	,! 37 ( $\Rightarrow E$ : 4,36)	i
$\omega[\mathbf{c}] \Rightarrow ((\oplus \mathbf{c})^I) \equiv (\mathbf{c} \ \infty)$	,! 38 ( $\Rightarrow I$ : 2,37)	i

$(\omega[\mathbf{c}] \Rightarrow ((\oplus\mathbf{c})^I) \equiv (\mathbf{c} \infty))$	,! 39 ((I: 38)	i
$\forall\mathbf{c} (\omega[\mathbf{c}] \Rightarrow ((\oplus\mathbf{c})^I) \equiv (\mathbf{c} \infty))$	! 40 ( $\forall$ I: 1,39)	i
$\square$		
<b>! 14.</b>		i
$\vdash \forall\mathbf{c} (\omega[\mathbf{c}] \Rightarrow \mathbf{1} (\oplus\mathbf{c}))$		i
$\mathbf{c}$	,! 1 (Prem)	i
$\omega[\mathbf{c}]$	,! 2 (Prem)	i
$\mathbf{x}, \mathbf{y}, \mathbf{z}$	,! 3 (Prem)	i
$(\oplus\mathbf{c})[\mathbf{x}, \mathbf{y}] \ \& \ (\oplus\mathbf{c})[\mathbf{z}, \mathbf{y}]$	,! 4 (Prem)	i
$(\oplus\mathbf{c})[\mathbf{x}, \mathbf{y}]$	,! 5 ( $\&$ E: 4)	i
$(\oplus\mathbf{c})[\mathbf{z}, \mathbf{y}]$	,! 6 ( $\&$ E: 4)	i
$\omega[\mathbf{c}] \ \& \ (\oplus\mathbf{c})[\mathbf{x}, \mathbf{y}]$	,! 7 ( $\&$ I: 2,5)	i
$(\omega[\mathbf{c}] \ \& \ (\oplus\mathbf{c})[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbf{x}+\mathbf{c}) = \mathbf{y})$	,! 8 ( $\forall$ E: P3)	i
$\omega[\mathbf{c}] \ \& \ (\oplus\mathbf{c})[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbf{x}+\mathbf{c}) = \mathbf{y}$	,! 9 ((E: 8)	i
$(\mathbf{x}+\mathbf{c}) = \mathbf{y}$	,! 10 ( $\Rightarrow$ E: 7,9)	i
$\omega[\mathbf{c}] \ \& \ (\oplus\mathbf{c})[\mathbf{z}, \mathbf{y}]$	,! 11 ( $\&$ I: 2,6)	i
$(\omega[\mathbf{c}] \ \& \ (\oplus\mathbf{c})[\mathbf{z}, \mathbf{y}] \Rightarrow (\mathbf{z}+\mathbf{c}) = \mathbf{y})$	,! 12 ( $\forall$ E: P3)	i
$\omega[\mathbf{c}] \ \& \ (\oplus\mathbf{c})[\mathbf{z}, \mathbf{y}] \Rightarrow (\mathbf{z}+\mathbf{c}) = \mathbf{y}$	,! 13 ((E: 12)	i
$(\mathbf{z}+\mathbf{c}) = \mathbf{y}$	,! 14 ( $\Rightarrow$ E: 11,13)	i
$(\mathbf{x}+\mathbf{c}) = (\mathbf{z}+\mathbf{c})$	,! 15 (=E: 10,14)	i
$(\mathbf{x}+\mathbf{c}) = (\mathbf{z}+\mathbf{c}) \Rightarrow \mathbf{x} = \mathbf{z}$	,! 16 ( $\forall$ E: C2.60)	i
$(\mathbf{x}+\mathbf{c}) = (\mathbf{z}+\mathbf{c}) \Rightarrow \mathbf{x} = \mathbf{z}$	,! 17 ((E: 16)	i
$\mathbf{x} = \mathbf{z}$	,! 18 ( $\Rightarrow$ E: 15,17)	i
$(\oplus\mathbf{c})[\mathbf{x}, \mathbf{y}] \ \& \ (\oplus\mathbf{c})[\mathbf{z}, \mathbf{y}] \Rightarrow \mathbf{x} = \mathbf{z}$	,! 19 ( $\Rightarrow$ I: 4,18)	i
$((\oplus\mathbf{c})[\mathbf{x}, \mathbf{y}] \ \& \ (\oplus\mathbf{c})[\mathbf{z}, \mathbf{y}] \Rightarrow \mathbf{x} = \mathbf{z})$	,! 20 ((I: 19)	i
$\forall\mathbf{x}\forall\mathbf{y}\forall\mathbf{z} ((\oplus\mathbf{c})[\mathbf{x}, \mathbf{y}] \ \& \ (\oplus\mathbf{c})[\mathbf{z}, \mathbf{y}] \Rightarrow \mathbf{x} = \mathbf{z})$	,! 21 ( $\forall$ I: 3,20)	i
$\mathbf{1} (\oplus\mathbf{c})$	,! 22 ( $\S$ I: III9.1,21)	

i

$\omega[c] \Rightarrow \mathbf{1} (\oplus c)$  ,! 23 ( $\Rightarrow$ I: 2,22) i

$( \omega[c] \Rightarrow \mathbf{1} (\oplus c) )$  ,! 24 ( $($ )I: 23) i

$\forall c ( \omega[c] \Rightarrow \mathbf{1} (\oplus c) )$  ! 25 ( $\forall$ I: 1,24) i

□

! 15. i

$\vdash \forall c ( \omega[c] \Rightarrow (\oplus c) \mathbf{1} (c \infty) )$  i

$c$  ,! 1 (Prem) i

$\omega[c]$  ,! 2 (Prem) i

$( \omega[c] \Rightarrow ((\oplus c)^I) \equiv (c \infty) )$  ,! 3 ( $\forall$ E: P13) i

$\omega[c] \Rightarrow ((\oplus c)^I) \equiv (c \infty)$  ,! 4 ( $($ )E: 3) i

$((\oplus c)^I) \equiv (c \infty)$  ,! 5 ( $\Rightarrow$ E: 2,4) i

$( \omega[c] \Rightarrow \mathbf{1} (\oplus c) )$  ,! 6 ( $\forall$ E: P14) i

$\omega[c] \Rightarrow \mathbf{1} (\oplus c)$  ,! 7 ( $($ )E: 6) i

$\mathbf{1} (\oplus c)$  ,! 8 ( $\Rightarrow$ E: 2,7) i

$((\oplus c)^I) \equiv (c \infty) \ \& \ \mathbf{1} (\oplus c)$  ,! 9 ( $\&$ I: 5,8) i

$(\oplus c) \mathbf{1} (c \infty)$  ,! 10 ( $\$$ I: III9.19,9) i

$\omega[c] \Rightarrow (\oplus c) \mathbf{1} (c \infty)$  ,! 11 ( $\Rightarrow$ I: 2,10) i

$( \omega[c] \Rightarrow (\oplus c) \mathbf{1} (c \infty) )$  ,! 12 ( $($ )I: 11) i

$\forall c ( \omega[c] \Rightarrow (\oplus c) \mathbf{1} (c \infty) )$  ! 13 ( $\forall$ I: 1,12) i

□

! 16. The finite numbers correspond to the infinite interval beginning at any finite number  $c$ . i

$\vdash \forall c ( \omega[c] \Rightarrow \omega \sim (c \infty) )$  i

$c$  ,! 1 (Prem) i

$\omega[c]$  ,! 2 (Prem) i

$( \omega[c] \Rightarrow (\oplus c) \mathbf{F} \omega )$  ,! 3 ( $\forall$ E: P11) i

$\omega[c] \Rightarrow (\oplus c) \mathbf{F} \omega$  ,! 4 ( $($ )E: 3) i

$(\oplus c) \mathbf{F} \omega$	,! 5 ( $\Rightarrow E: 2,4$ )	i
$(\omega[c] \Rightarrow (\oplus c) \mathbf{1} (c \infty))$	,! 6 ( $\forall E: P15$ )	i
$\omega[c] \Rightarrow (\oplus c) \mathbf{1} (c \infty)$	,! 7 ( $(\Rightarrow) E: 6$ )	i
$(\oplus c) \mathbf{1} (c \infty)$	,! 8 ( $\Rightarrow E: 2,7$ )	i
$(\oplus c) \mathbf{F} \omega \ \& \ (\oplus c) \mathbf{1} (c \infty)$	,! 9 ( $\& I: 5,8$ )	i
$( (\oplus c) \mathbf{F} \omega \ \& \ (\oplus c) \mathbf{1} (c \infty) )$	,! 10 ( $(\Rightarrow) I: 9$ )	i
$\exists R ( R \mathbf{F} \omega \ \& \ R \mathbf{1} (c \infty) )$	,! 11 ( $\exists I: 10$ )	i
$\omega \sim (c \infty)$	,! 12 ( $\S I: III13.1,11$ )	i
$\omega[c] \Rightarrow \omega \sim (c \infty)$	,! 13 ( $\Rightarrow I: 2,12$ )	i
$(\omega[c] \Rightarrow \omega \sim (c \infty))$	,! 14 ( $(\Rightarrow) I: 13$ )	i
$\forall c ( \omega[c] \Rightarrow \omega \sim (c \infty) )$	! 15 ( $\forall I: 1,14$ )	i

□

! 17.  $(\ominus c)$  will represent dyadic subtraction, the subtracting of  $c$ .

$\mathbb{D} \ \ominus ; (\ominus c) ; ; (((\ominus \diamond c) \lceil (c \infty)) \lfloor \omega)$

! 18.

$\vdash \forall c \forall x \forall y ( (\ominus c)[x,y] \Leftrightarrow (x-c) = y )$

$c, x, y$  ,! 1 (Prem) i

$( (\ominus \diamond c)[x,y] \Leftrightarrow \ominus[x,c,y] )$  ,! 2 ( $\forall E: III17.2$ ) i

$(\ominus \diamond c)[x,y] \Leftrightarrow \ominus[x,c,y]$  ,! 3 ( $(\Rightarrow) E: 2$ ) i

$( ((\ominus \diamond c) \lceil (c \infty))[x,y] \Leftrightarrow (\ominus \diamond c)[x,y] \ \& \ (c \infty)[x] )$   
, ! 4 ( $\forall E: III7.2$ ) i

$((\ominus \diamond c) \lceil (c \infty))[x,y] \Leftrightarrow (\ominus \diamond c)[x,y] \ \& \ (c \infty)[x]$   
, ! 5 ( $(\Rightarrow) E: 4$ ) i

$( (((\ominus \diamond c) \lceil (c \infty)) \lfloor \omega)[x,y] \Leftrightarrow ((\ominus \diamond c) \lceil (c \infty))[x,y] \ \& \ \omega[y] )$   
, ! 6 ( $\forall E: III7.50$ ) i

$( ((\ominus \diamond c) \lceil (c \infty)) \lfloor \omega)[x,y] \Leftrightarrow ((\ominus \diamond c) \lceil (c \infty))[x,y] \ \& \ \omega[y]$   
, ! 7 ( $(\Rightarrow) E: 6$ ) i

$(\ominus c)[x,y] \Leftrightarrow ((\ominus \diamond c) \lceil (c \infty))[x,y] \ \& \ \omega[y]$   
, ! 8 ( $\mathbb{D} I: P17,7$ ) i

$( (c \infty)[x] \Leftrightarrow \leq[c,x] )$	,! 9 ( $\forall E$ : C9.2)	i
$(c \infty)[x] \Leftrightarrow \leq[c,x]$	,! 10 ( $(\Rightarrow)E$ : 9)	i
$(\Theta c)[x,y]$	,! 11 (Prem)	i
$(\Theta c)[x,y] \Rightarrow ((\Theta \diamond c) \lceil (c \infty))[x,y] \ \& \ \omega[y]$	,! 12 ( $(\Leftrightarrow)E$ : 8)	i
$((\Theta \diamond c) \lceil (c \infty))[x,y] \ \& \ \omega[y]$	,! 13 ( $(\Rightarrow)E$ : 11,12)	i
$((\Theta \diamond c) \lceil (c \infty))[x,y]$	,! 14 ( $(\&E)$ : 13)	i
$\omega[y]$	,! 15 ( $(\&E)$ : 13)	i
$((\Theta \diamond c) \lceil (c \infty))[x,y] \Rightarrow (\Theta \diamond c)[x,y] \ \& \ (c \infty)[x]$	,! 16 ( $(\Leftrightarrow)E$ : 5)	i
$(\Theta \diamond c)[x,y] \ \& \ (c \infty)[x]$	,! 17 ( $(\Rightarrow)E$ : 14,16)	i
$(\Theta \diamond c)[x,y]$	,! 18 ( $(\&E)$ : 17)	i
$(c \infty)[x]$	,! 19 ( $(\&E)$ : 17)	i
$(c \infty)[x] \Rightarrow \leq[c,x]$	,! 20 ( $(\Leftrightarrow)E$ : 10)	i
$\leq[c,x]$	,! 21 ( $(\Rightarrow)E$ : 19,20)	i
$\omega[y] \ \& \ \leq[c,x]$	,! 22 ( $(\&E)$ : 15,21)	i
$(\Theta \diamond c)[x,y] \Rightarrow \Theta[x,c,y]$	,! 23 ( $(\Leftrightarrow)E$ : 3)	i
$\Theta[x,c,y]$	,! 24 ( $(\Rightarrow)E$ : 18,23)	i
$\omega[y] \ \& \ \leq[c,x] \ \& \ \Theta[x,c,y]$	,! 25 ( $(\&I)$ : 22,24)	i
$( \omega[y] \ \& \ \leq[c,x] \ \& \ \Theta[x,c,y] \Rightarrow (x-c) = y )$	,! 26 ( $(\forall E)$ : C5.12)	i
$\omega[y] \ \& \ \leq[c,x] \ \& \ \Theta[x,c,y] \Rightarrow (x-c) = y$ ,	! 27 ( $(\Rightarrow)E$ : 26)	i
$(x-c) = y$	,! 28 ( $(\Rightarrow)E$ : 25,27)	i
$(\Theta c)[x,y] \Rightarrow (x-c) = y$	,! 29 ( $(\Rightarrow I)$ : 11,28)	i
$(x-c) = y$	,! 30 (Prem)	i
$\leq[c,x]$	,! 31 ( $(\mathbb{T}E)$ : C5.7,30)	i
$( \leq[c,x] \Rightarrow \Theta[x,c,(x-c)] )$	,! 32 ( $(\forall E)$ : C5.10)	i
$\leq[c,x] \Rightarrow \Theta[x,c,(x-c)]$	,! 33 ( $(\Rightarrow)E$ : 32)	i

$\Theta[\mathbf{x}, \mathbf{c}, (\mathbf{x}-\mathbf{c})]$	,!	34	( $\Rightarrow$ E: 31,33)	i
$\Theta[\mathbf{x}, \mathbf{c}, \mathbf{y}]$	,!	35	( $=$ E: 30,34)	i
$\leq[\mathbf{c}, \mathbf{x}] \Rightarrow (\mathbf{c} \infty)[\mathbf{x}]$	,!	36	( $\Leftrightarrow$ E: 10)	i
$(\mathbf{c} \infty)[\mathbf{x}]$	,!	37	( $\Rightarrow$ E: 31,36)	i
$\Theta[\mathbf{x}, \mathbf{c}, \mathbf{y}] \Rightarrow (\Theta \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}]$	,!	38	( $\Leftrightarrow$ E: 3)	i
$(\Theta \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}]$	,!	39	( $\Rightarrow$ E: 35,38)	i
$(\Theta \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}] \ \& \ (\mathbf{c} \infty)[\mathbf{x}]$	,!	40	( $\&$ I: 37,39)	i
$(\Theta \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}] \ \& \ (\mathbf{c} \infty)[\mathbf{x}] \Rightarrow ((\Theta \diamond \mathbf{c}) \lceil (\mathbf{c} \infty))[\mathbf{x}, \mathbf{y}]$	,!	41	( $\Leftrightarrow$ E: 5)	i
$((\Theta \diamond \mathbf{c}) \lceil (\mathbf{c} \infty))[\mathbf{x}, \mathbf{y}]$	,!	42	( $\Rightarrow$ E: 40,41)	i
$( (\mathbf{x}-\mathbf{c}) = \mathbf{y} \Rightarrow \omega[\mathbf{y}] )$	,!	43	( $\forall$ E: C5.9)	i
$(\mathbf{x}-\mathbf{c}) = \mathbf{y} \Rightarrow \omega[\mathbf{y}]$	,!	44	( $()$ E: 43)	i
$\omega[\mathbf{y}]$	,!	45	( $\Rightarrow$ E: 30,44)	i
$((\Theta \diamond \mathbf{c}) \lceil (\mathbf{c} \infty))[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{y}]$	,!	46	( $\&$ I: 42,45)	i
$((\Theta \diamond \mathbf{c}) \lceil (\mathbf{c} \infty))[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{y}] \Rightarrow (\Theta \mathbf{c})[\mathbf{x}, \mathbf{y}]$	,!	47	( $\Leftrightarrow$ E: 8)	i
$(\Theta \mathbf{c})[\mathbf{x}, \mathbf{y}]$	,!	48	( $\Rightarrow$ E: 46,47)	i
$(\mathbf{x}-\mathbf{c}) = \mathbf{y} \Rightarrow (\Theta \mathbf{c})[\mathbf{x}, \mathbf{y}]$	,!	49	( $\Rightarrow$ I: 30,48)	i
$(\Theta \mathbf{c})[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\mathbf{x}-\mathbf{c}) = \mathbf{y}$	,!	50	( $\Leftrightarrow$ I: 29,49)	i
$( (\Theta \mathbf{c})[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\mathbf{x}-\mathbf{c}) = \mathbf{y} )$	,!	51	( $()$ I: 50)	i
$\forall \mathbf{c} \forall \mathbf{x} \forall \mathbf{y} ( (\Theta \mathbf{c})[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\mathbf{x}-\mathbf{c}) = \mathbf{y} )$	!	52	( $\forall$ I: 1,51)	i

□

! 19.

$\vdash \forall \mathbf{c} \forall \mathbf{x} \forall \mathbf{y} ( (\Theta \mathbf{c})[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbf{x}-\mathbf{c}) = \mathbf{y} )$				i
$\mathbf{c}, \mathbf{x}, \mathbf{y}$	,!	1	(Prem)	i
$( (\Theta \mathbf{c})[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\mathbf{x}-\mathbf{c}) = \mathbf{y} )$	,!	2	( $\forall$ E: P18)	i
$(\Theta \mathbf{c})[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\mathbf{x}-\mathbf{c}) = \mathbf{y}$	,!	3	( $()$ E: 2)	i
$(\Theta \mathbf{c})[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbf{x}-\mathbf{c}) = \mathbf{y}$	,!	4	( $\Leftrightarrow$ E: 3)	i

$( (\Theta c)[x,y] \Rightarrow (x-c) = y )$	,! 5 ((I: 4)	i
$\forall c \forall x \forall y ( (\Theta c)[x,y] \Rightarrow (x-c) = y )$	! 6 ( $\forall$ I: 1,5)	i
$\square$		
! 20.		i
$\vdash \forall c \forall x \forall y ( (x-c) = y \Rightarrow (\Theta c)[x,y] )$		i
$c, x, y$	,! 1 (Prem)	i
$( (\Theta c)[x,y] \Leftrightarrow (x-c) = y )$	,! 2 ( $\forall$ E: P18)	i
$(\Theta c)[x,y] \Leftrightarrow (x-c) = y$	,! 3 ((E: 2)	i
$(x-c) = y \Rightarrow (\Theta c)[x,y]$	,! 4 ( $\Leftrightarrow$ E: 3)	i
$( (x-c) = y \Rightarrow (\Theta c)[x,y] )$	,! 5 ((I: 4)	i
$\forall c \forall x \forall y ( (x-c) = y \Rightarrow (\Theta c)[x,y] )$	! 6 ( $\forall$ I: 1,5)	i
$\square$		
! 21.		i
$\vdash \forall c \forall x ( \leq[c,x] \Rightarrow (\Theta c)[x,(x-c)] )$		i
$c, x$	,! 1 (Prem)	i
$\leq[c,x]$	,! 2 (Prem)	i
$(x-c) = (x-c)$	,! 3 (=I; (x-c): C5.7,2)	i
$( (x-c) = (x-c) \Rightarrow (\Theta c)[x,(x-c)] )$	,! 4 ( $\forall$ E: P20; (x-c): C5.7,2)	i
$(x-c) = (x-c) \Rightarrow (\Theta c)[x,(x-c)]$	,! 5 ((E: 4)	i
$(\Theta c)[x,(x-c)]$	,! 6 ( $\Rightarrow$ E: 3,5)	i
$\leq[c,x] \Rightarrow (\Theta c)[x,(x-c)]$	,! 7 ( $\Rightarrow$ I: 2,6)	i
$( \leq[c,x] \Rightarrow (\Theta c)[x,(x-c)] )$	,! 8 ((I: 7)	i
$\forall c \forall x ( \leq[c,x] \Rightarrow (\Theta c)[x,(x-c)] )$	! 9 ( $\forall$ I: 1,8)	i
$\square$		
! 22.		i
$\vdash \forall c \forall x \forall y ( (\Theta c)[x,y] \Rightarrow \omega[c] \ \& \ \omega[x] \ \& \ \omega[y] )$		i
$c, x, y$	,! 1 (Prem)	i
$(\Theta c)[x,y]$	,! 2 (Prem)	i

$( (\ominus c)[x, y] \Rightarrow (x - c) = y )$	,! 3 ( $\forall E$ : P19)	i
$(\ominus c)[x, y] \Rightarrow (x - c) = y$	,! 4 ( $()E$ : 3)	i
$(x - c) = y$	,! 5 ( $\Rightarrow E$ : 2,4)	i
$\leq[c, x]$	,! 6 ( $\mathbb{T}E$ : C5.7,5)	i
$( \leq[c, x] \Rightarrow \omega[c] \ \& \ \omega[x] )$	,! 7 ( $\forall E$ : C3.5)	i
$\leq[c, x] \Rightarrow \omega[c] \ \& \ \omega[x]$	,! 8 ( $()E$ : 7)	i
$\omega[c] \ \& \ \omega[x]$	,! 9 ( $\Rightarrow E$ : 6,8)	i
$( \leq[c, x] \Rightarrow \omega[(x - c)] )$	,! 10 ( $\forall E$ : C5.8)	i
$\leq[c, x] \Rightarrow \omega[(x - c)]$	,! 11 ( $()E$ : 10)	i
$\omega[(x - c)]$	,! 12 ( $\Rightarrow E$ : 6,11)	i
$\omega[y]$	,! 13 ( $=E$ : 5,12)	i
$\omega[c] \ \& \ \omega[x] \ \& \ \omega[y]$	,! 14 ( $\&I$ : 9,13)	i
$(\ominus c)[x, y] \Rightarrow \omega[c] \ \& \ \omega[x] \ \& \ \omega[y]$	,! 15 ( $\Rightarrow I$ : 2,14)	i
$( (\ominus c)[x, y] \Rightarrow \omega[c] \ \& \ \omega[x] \ \& \ \omega[y] )$	,! 16 ( $()E$ : 15)	i
$\forall c \forall x \forall y ( (\ominus c)[x, y] \Rightarrow \omega[c] \ \& \ \omega[x] \ \& \ \omega[y] )$	! 17 ( $\forall I$ : 1,16)	i

□

! P23 and P24 establish that adding  $c$  and subtracting  $c$  are inverses (when  $c$  is a finite number).

! 23.

$\vdash \forall c ( \omega[c] \Rightarrow ((\ominus c)^*) \equiv (\oplus c) )$		i
$c$	,! 1 (Prem)	i
$\omega[c]$	,! 2 (Prem)	i
$x, y$	,! 3 (Prem)	i
$( ((\ominus c)^*)[x, y] \Leftrightarrow (\ominus c)[y, x] )$	,! 4 ( $\forall E$ : III3.2)	i
$((\ominus c)^*)[x, y] \Leftrightarrow (\ominus c)[y, x]$	,! 5 ( $()E$ : 4)	i
$( (\ominus c)[y, x] \Leftrightarrow (y - c) = x )$	,! 6 ( $\forall E$ : P18)	i
$(\ominus c)[y, x] \Leftrightarrow (y - c) = x$	,! 7 ( $()E$ : 6)	i

$(\mathbf{y} - \mathbf{c}) = \mathbf{x} \Leftrightarrow (\mathbf{x} + \mathbf{c}) = \mathbf{y}$  ,! 8 ( $\forall E$ : C6.6) i  
 $(\mathbf{y} - \mathbf{c}) = \mathbf{x} \Leftrightarrow (\mathbf{x} + \mathbf{c}) = \mathbf{y}$  ,! 9 ( $()E$ : 8) i  
 $(\omega[\mathbf{c}] \ \& \ (\oplus\mathbf{c})[\mathbf{x},\mathbf{y}] \Leftrightarrow (\mathbf{x} + \mathbf{c}) = \mathbf{y})$  ,! 10 ( $\forall E$ : P2) i  
 $\omega[\mathbf{c}] \ \& \ (\oplus\mathbf{c})[\mathbf{x},\mathbf{y}] \Leftrightarrow (\mathbf{x} + \mathbf{c}) = \mathbf{y}$  ,! 11 ( $()E$ : 10) i  
 $((\ominus\mathbf{c})^*)[\mathbf{x},\mathbf{y}]$  ,! 12 (Prem) i  
 $((\ominus\mathbf{c})^*)[\mathbf{x},\mathbf{y}] \Rightarrow (\ominus\mathbf{c})[\mathbf{y},\mathbf{x}]$  ,! 13 ( $\Leftrightarrow E$ : 5) i  
 $(\ominus\mathbf{c})[\mathbf{y},\mathbf{x}]$  ,! 14 ( $\Rightarrow E$ : 12,13) i  
 $(\ominus\mathbf{c})[\mathbf{y},\mathbf{x}] \Rightarrow (\mathbf{y} - \mathbf{c}) = \mathbf{x}$  ,! 15 ( $\Leftrightarrow E$ : 7) i  
 $(\mathbf{y} - \mathbf{c}) = \mathbf{x}$  ,! 16 ( $\Rightarrow E$ : 14,15) i  
 $(\mathbf{y} - \mathbf{c}) = \mathbf{x} \Rightarrow (\mathbf{x} + \mathbf{c}) = \mathbf{y}$  ,! 17 ( $\Leftrightarrow E$ : 9) i  
 $(\mathbf{x} + \mathbf{c}) = \mathbf{y}$  ,! 18 ( $\Rightarrow E$ : 16,17) i  
 $(\mathbf{x} + \mathbf{c}) = \mathbf{y} \Rightarrow \omega[\mathbf{c}] \ \& \ (\oplus\mathbf{c})[\mathbf{x},\mathbf{y}]$  ,! 19 ( $\Leftrightarrow E$ : 11) i  
 $\omega[\mathbf{c}] \ \& \ (\oplus\mathbf{c})[\mathbf{x},\mathbf{y}]$  ,! 20 ( $\Rightarrow E$ : 18,19) i  
 $(\oplus\mathbf{c})[\mathbf{x},\mathbf{y}]$  ,! 21 ( $\& E$ : 20) i  
 $((\ominus\mathbf{c})^*)[\mathbf{x},\mathbf{y}] \Rightarrow (\oplus\mathbf{c})[\mathbf{x},\mathbf{y}]$  ,! 22 ( $\Rightarrow I$ : 12,21) i  
 $(\oplus\mathbf{c})[\mathbf{x},\mathbf{y}]$  ,! 23 (Prem) i  
 $\omega[\mathbf{c}] \ \& \ (\oplus\mathbf{c})[\mathbf{x},\mathbf{y}]$  ,! 24 ( $\& I$ : 2,23) i  
 $\omega[\mathbf{c}] \ \& \ (\oplus\mathbf{c})[\mathbf{x},\mathbf{y}] \Rightarrow (\mathbf{x} + \mathbf{c}) = \mathbf{y}$  ,! 25 ( $\Leftrightarrow E$ : 11) i  
 $(\mathbf{x} + \mathbf{c}) = \mathbf{y}$  ,! 26 ( $\Rightarrow E$ : 24,25) i  
 $(\mathbf{x} + \mathbf{c}) = \mathbf{y} \Rightarrow (\mathbf{y} - \mathbf{c}) = \mathbf{x}$  ,! 27 ( $\Leftrightarrow E$ : 9) i  
 $(\mathbf{y} - \mathbf{c}) = \mathbf{x}$  ,! 28 ( $\Rightarrow E$ : 26,27) i  
 $(\mathbf{y} - \mathbf{c}) = \mathbf{x} \Rightarrow (\ominus\mathbf{c})[\mathbf{y},\mathbf{x}]$  ,! 29 ( $\Leftrightarrow E$ : 7) i  
 $(\ominus\mathbf{c})[\mathbf{y},\mathbf{x}]$  ,! 30 ( $\Rightarrow E$ : 28,29) i  
 $(\ominus\mathbf{c})[\mathbf{y},\mathbf{x}] \Rightarrow ((\ominus\mathbf{c})^*)[\mathbf{x},\mathbf{y}]$  ,! 31 ( $\Leftrightarrow E$ : 5) i  
 $((\ominus\mathbf{c})^*)[\mathbf{x},\mathbf{y}]$  ,! 32 ( $\Rightarrow E$ : 30,31) i  
 $(\oplus\mathbf{c})[\mathbf{x},\mathbf{y}] \Rightarrow ((\ominus\mathbf{c})^*)[\mathbf{x},\mathbf{y}]$  ,! 33 ( $\Rightarrow I$ : 23,32) i

$((\Theta c)^*)[x, y] \Leftrightarrow (\oplus c)[x, y]$  ,! 34 ( $\Leftrightarrow$ I: 22,33) i  
 $( ((\Theta c)^*)[x, y] \Leftrightarrow (\oplus c)[x, y] )$  ,! 35 ( $(\ )$ I: 34) i  
 $\forall x \forall y ( ((\Theta c)^*)[x, y] \Leftrightarrow (\oplus c)[x, y] )$  ,! 36 ( $\forall$ I: 3,35) i  
 $((\Theta c)^*) \equiv (\oplus c)$  ,! 37 ( $\S$ I: III1.5,36) i  
 $\omega[c] \Rightarrow ((\Theta c)^*) \equiv (\oplus c)$  ,! 38 ( $\Rightarrow$ I: 2,37) i  
 $( \omega[c] \Rightarrow ((\Theta c)^*) \equiv (\oplus c) )$  ,! 39 ( $(\ )$ I: 38) i  
 $\forall c ( \omega[c] \Rightarrow ((\Theta c)^*) \equiv (\oplus c) )$  ! 40 ( $\forall$ I: 1,39) i  
 $\square$

! 24.

$\vdash \forall c ( \omega[c] \Rightarrow ((\oplus c)^*) \equiv (\Theta c) )$  i  
 $c$  ,! 1 (Prem) i  
 $\omega[c]$  ,! 2 (Prem) i  
 $( \omega[c] \Rightarrow ((\Theta c)^*) \equiv (\oplus c) )$  ,! 3 ( $\forall$ E: P23) i  
 $\omega[c] \Rightarrow ((\Theta c)^*) \equiv (\oplus c)$  ,! 4 ( $(\ )$ E: 3) i  
 $((\Theta c)^*) \equiv (\oplus c)$  ,! 5 ( $\Rightarrow$ E: 2,4) i  
 $( ((\Theta c)^*) \equiv (\oplus c) \Rightarrow ((\oplus c)^*) \equiv (\Theta c) )$  ,! 6 ( $\forall$ E: III3.23) i  
 $((\Theta c)^*) \equiv (\oplus c) \Rightarrow ((\oplus c)^*) \equiv (\Theta c)$  ,! 7 ( $(\ )$ E: 6) i  
 $((\oplus c)^*) \equiv (\Theta c)$  ,! 8 ( $\Rightarrow$ E: 5,7) i  
 $\omega[c] \Rightarrow ((\oplus c)^*) \equiv (\Theta c)$  ,! 9 ( $\Rightarrow$ I: 2,8) i  
 $( \omega[c] \Rightarrow ((\oplus c)^*) \equiv (\Theta c) )$  ,! 10 ( $(\ )$ I: 9) i  
 $\forall c ( \omega[c] \Rightarrow ((\oplus c)^*) \equiv (\Theta c) )$  ! 11 ( $\forall$ I: 1,10) i  
 $\square$

! 25. Subtracting 0 is the identity on  $\omega$ .

$\vdash (\Theta 0) \equiv (\mathbb{I}\omega)$  i  
 $( \omega[0] \Rightarrow ((\oplus 0)^*) \equiv (\Theta 0) )$  ,! 1 ( $\forall$ E: P24) i  
 $\omega[0] \Rightarrow ((\oplus 0)^*) \equiv (\Theta 0)$  ,! 2 ( $(\ )$ E: 1) i

$((\oplus 0)^*) \equiv (\ominus 0)$  ,! 3 ( $\Rightarrow$ E:  $\omega 0, 2$ ) i  
 $((\oplus 0) \equiv (\mathbb{I}\omega) \Rightarrow ((\oplus 0)^*) \equiv ((\mathbb{I}\omega)^*))$  ,! 4 ( $\forall$ E: III3.20) i  
 $(\oplus 0) \equiv (\mathbb{I}\omega) \Rightarrow ((\oplus 0)^*) \equiv ((\mathbb{I}\omega)^*)$  ,! 5 ( $()$ E: 4) i  
 $((\oplus 0)^*) \equiv ((\mathbb{I}\omega)^*)$  ,! 6 ( $\Rightarrow$ E: P7,5) i  
 $((\oplus 0)^*) \equiv (\ominus 0) \ \& \ ((\oplus 0)^*) \equiv ((\mathbb{I}\omega)^*)$  ,! 7 ( $\&$ I: 3,6) i  
 $((\oplus 0)^*) \equiv (\ominus 0) \ \& \ ((\oplus 0)^*) \equiv ((\mathbb{I}\omega)^*) \Rightarrow (\ominus 0) \equiv ((\mathbb{I}\omega)^*)$  ,! 8 ( $\forall$ E: III1.18) i  
 $((\oplus 0)^*) \equiv (\ominus 0) \ \& \ ((\oplus 0)^*) \equiv ((\mathbb{I}\omega)^*) \Rightarrow (\ominus 0) \equiv ((\mathbb{I}\omega)^*)$  ,! 9 ( $()$ E: 8) i  
 $(\ominus 0) \equiv ((\mathbb{I}\omega)^*)$  ,! 10 ( $\Rightarrow$ E: 7,9) i  
 $((\mathbb{I}\omega)^*) \equiv (\mathbb{I}\omega)$  ,! 11 ( $\forall$ E: III11.23) i  
 $(\ominus 0) \equiv ((\mathbb{I}\omega)^*) \ \& \ ((\mathbb{I}\omega)^*) \equiv (\mathbb{I}\omega)$  ,! 12 ( $\&$ I: 10,11) i  
 $((\ominus 0) \equiv ((\mathbb{I}\omega)^*) \ \& \ ((\mathbb{I}\omega)^*) \equiv (\mathbb{I}\omega) \Rightarrow (\ominus 0) \equiv (\mathbb{I}\omega)$  ,! 13 ( $\forall$ E: III1.15) i  
 $(\ominus 0) \equiv ((\mathbb{I}\omega)^*) \ \& \ ((\mathbb{I}\omega)^*) \equiv (\mathbb{I}\omega) \Rightarrow (\ominus 0) \equiv (\mathbb{I}\omega)$  ,! 14 ( $()$ E: 13) i  
 $(\ominus 0) \equiv (\mathbb{I}\omega)$  ! 15 ( $\Rightarrow$ E: 12,14) i

□

! 26. Subtracting 1 is the predecession on  $\omega$ . i

$\vdash (\ominus 1) \equiv ((\sigma \lceil \omega)^*)$  i  
 $(\omega[1] \Rightarrow ((\oplus 1)^*) \equiv (\ominus 1))$  ,! 1 ( $\forall$ E: P24) i  
 $\omega[1] \Rightarrow ((\oplus 1)^*) \equiv (\ominus 1)$  ,! 2 ( $()$ E: 1) i  
 $((\oplus 1)^*) \equiv (\ominus 1)$  ,! 3 ( $\Rightarrow$ E: IV9.2,2) i  
 $((\oplus 1) \equiv (\sigma \lceil \omega) \Rightarrow ((\oplus 1)^*) \equiv ((\sigma \lceil \omega)^*))$  ,! 4 ( $\forall$ E: III3.20) i  
 $(\oplus 1) \equiv (\sigma \lceil \omega) \Rightarrow ((\oplus 1)^*) \equiv ((\sigma \lceil \omega)^*)$  ,! 5 ( $()$ E: 4) i  
 $((\oplus 1)^*) \equiv ((\sigma \lceil \omega)^*)$  ,! 6 ( $\Rightarrow$ E: P8,5) i  
 $((\oplus 1)^*) \equiv (\ominus 1) \ \& \ ((\oplus 1)^*) \equiv ((\sigma \lceil \omega)^*)$  ,! 7 ( $\&$ I: 3,6) i

$( ((\oplus 1)^*) \equiv (\ominus 1) \ \& \ ((\oplus 1)^*) \equiv ((\sigma \lceil \omega)^*) \Rightarrow (\ominus 1) \equiv ((\sigma \lceil \omega)^*) )$   
, ! 8 ( $\forall E$ : III1.18) i

$((\oplus 1)^*) \equiv (\ominus 1) \ \& \ ((\oplus 1)^*) \equiv ((\sigma \lceil \omega)^*) \Rightarrow (\ominus 1) \equiv ((\sigma \lceil \omega)^*)$   
, ! 9 ( $()E$ : 8) i

$(\ominus 1) \equiv ((\sigma \lceil \omega)^*)$   
, ! 10 ( $\Rightarrow E$ : 7,9) i

□

! 27. i

$\vdash \forall c ( \omega[c] \Rightarrow f(\ominus c) )$  i

$c$  , ! 1 (Prem) i

$\omega[c]$  , ! 2 (Prem) i

$( \omega[c] \Rightarrow \mathbf{1}(\oplus c) )$  , ! 3 ( $\forall E$ : P14) i

$\omega[c] \Rightarrow \mathbf{1}(\oplus c)$  , ! 4 ( $()E$ : 3) i

$\mathbf{1}(\oplus c)$  , ! 5 ( $\Rightarrow E$ : 2,4) i

$( \omega[c] \Rightarrow ((\ominus c)^*) \equiv (\oplus c) )$  , ! 6 ( $\forall E$ : P23) i

$\omega[c] \Rightarrow ((\ominus c)^*) \equiv (\oplus c)$  , ! 7 ( $()E$ : 6) i

$((\ominus c)^*) \equiv (\oplus c)$  , ! 8 ( $\Rightarrow E$ : 2,7) i

$\mathbf{1}(\oplus c) \ \& \ ((\ominus c)^*) \equiv (\oplus c)$  , ! 9 ( $\&I$ : 5,8) i

$( \mathbf{1}(\oplus c) \ \& \ ((\ominus c)^*) \equiv (\oplus c) \Rightarrow f(\ominus c) )$   
, ! 10 ( $\forall E$ : III9.14) i

$\mathbf{1}(\oplus c) \ \& \ ((\ominus c)^*) \equiv (\oplus c) \Rightarrow f(\ominus c)$  , ! 11 ( $()E$ : 10) i

$f(\ominus c)$  , ! 12 ( $\Rightarrow E$ : 9,11) i

$\omega[c] \Rightarrow f(\ominus c)$  , ! 13 ( $\Rightarrow I$ : 2,12) i

$( \omega[c] \Rightarrow f(\ominus c) )$  , ! 14 ( $()I$ : 13) i

$\forall c ( \omega[c] \Rightarrow f(\ominus c) )$  ! 15 ( $\forall I$ : 1,14) i

□

! 28. i

$\vdash \forall c \forall x ( \leq[c,x] \Rightarrow ((\ominus c) \dot{\prime} x) = (x-c) )$  i

$c, x$  , ! 1 (Prem) i

$\leq[c,x]$  , ! 2 (Prem) i

$( \leq[c, \mathbf{x}] \Rightarrow \omega[c] )$	,! 3 ( $\forall E$ : C3.6)	i
$\leq[c, \mathbf{x}] \Rightarrow \omega[c]$	,! 4 ( $(\ )E$ : 3)	i
$\omega[c]$	,! 5 ( $\Rightarrow E$ : 2,4)	i
$( \omega[c] \Rightarrow \mathbf{f}(\Theta c) )$	,! 6 ( $\forall E$ : P10)	i
$\omega[c] \Rightarrow \mathbf{f}(\Theta c)$	,! 7 ( $(\ )E$ : 6)	i
$\mathbf{f}(\Theta c)$	,! 8 ( $\Rightarrow E$ : 5,7)	i
$( \leq[c, \mathbf{x}] \Rightarrow (\Theta c)[\mathbf{x}, (\mathbf{x}-c)] )$	,! 9 ( $\forall E$ : P21)	i
$\leq[c, \mathbf{x}] \Rightarrow (\Theta c)[\mathbf{x}, (\mathbf{x}-c)]$	,! 10 ( $(\ )E$ : 9)	i
$(\Theta c)[\mathbf{x}, (\mathbf{x}-c)]$	,! 11 ( $\Rightarrow E$ : 2,10)	i
$\mathbf{f}(\Theta c) \ \& \ (\Theta c)[\mathbf{x}, (\mathbf{x}-c)]$	,! 12 ( $\&I$ : 8,11)	i
$( \mathbf{f}(\Theta c) \ \& \ (\Theta c)[\mathbf{x}, (\mathbf{x}-c)] \Rightarrow ((\Theta c) \acute{x}) = (\mathbf{x}-c) )$	,! 13 ( $\forall E$ : III8.22; ( $\mathbf{x}-c$ ): C5.7,2)	i
$\mathbf{f}(\Theta c) \ \& \ (\Theta c)[\mathbf{x}, (\mathbf{x}-c)] \Rightarrow ((\Theta c) \acute{x}) = (\mathbf{x}-c)$	,! 14 ( $(\ )E$ : 13)	i
$((\Theta c) \acute{x}) = (\mathbf{x}-c)$	,! 15 ( $\Rightarrow E$ : 12,14)	i
$\leq[c, \mathbf{x}] \Rightarrow ((\Theta c) \acute{x}) = (\mathbf{x}-c)$	,! 16 ( $\Rightarrow I$ : 2,15)	i
$( \leq[c, \mathbf{x}] \Rightarrow ((\Theta c) \acute{x}) = (\mathbf{x}-c) )$	,! 17 ( $(\ )I$ : 16)	i
$\forall c \forall \mathbf{x} ( \leq[c, \mathbf{x}] \Rightarrow ((\Theta c) \acute{x}) = (\mathbf{x}-c) )$	! 18 ( $\forall I$ : 1,17)	i

□

! 29.

$\vdash \forall c ( \omega[c] \Rightarrow ((\Theta c)^D) \equiv (c \infty) )$		i
$c$	,! 1 (Prem)	i
$\omega[c]$	,! 2 (Prem)	i
$( \omega[c] \Rightarrow ((\Theta c)^I) \equiv (c \infty) )$	,! 3 ( $\forall E$ : P13)	i
$\omega[c] \Rightarrow ((\Theta c)^I) \equiv (c \infty)$	,! 4 ( $(\ )E$ : 3)	i
$((\Theta c)^I) \equiv (c \infty)$	,! 5 ( $\Rightarrow E$ : 2,4)	i
$( ((\Theta c)^I) \equiv (c \infty) \Rightarrow (((\Theta c)^*)^D) \equiv (c \infty) )$	,! 6 ( $\forall E$ : III6.17)	i

$((\oplus c)^I) \equiv (c \infty) \Rightarrow (((\oplus c)^*)^D) \equiv (c \infty)$	, ! 7 (( )E: 6)	i
$((\oplus c)^*)^D) \equiv (c \infty)$	, ! 8 ( $\Rightarrow$ E: 4,7)	i
$(\omega[c] \Rightarrow ((\oplus c)^*) \equiv (\ominus c) )$	, ! 9 ( $\forall$ E: P24)	i
$\omega[c] \Rightarrow ((\oplus c)^*) \equiv (\ominus c)$	, ! 10 (( )E: 9)	i
$((\oplus c)^*) \equiv (\ominus c)$	, ! 11 ( $\Rightarrow$ E: 2,10)	i
$((\oplus c)^*)^D) \equiv (c \infty) \ \& \ ((\oplus c)^*) \equiv (\ominus c)$	, ! 12 ( $\&$ I: 5,11)	i
$( (((\oplus c)^*)^D) \equiv (c \infty) \ \& \ ((\oplus c)^*) \equiv (\ominus c) \Rightarrow ((\ominus c)^D) \equiv (c \infty) )$	, ! 13 ( $\forall$ E: III5.16)	i
$((\oplus c)^*)^D) \equiv (c \infty) \ \& \ ((\oplus c)^*) \equiv (\ominus c) \Rightarrow ((\ominus c)^D) \equiv (c \infty)$	, ! 14 (( )E: 13)	i
$((\ominus c)^D) \equiv (c \infty)$	, ! 15 ( $\Rightarrow$ E: 12,14)	i
$\omega[c] \Rightarrow ((\ominus c)^D) \equiv (c \infty)$	, ! 16 ( $\Rightarrow$ I: 2,15)	i
$( \omega[c] \Rightarrow ((\ominus c)^D) \equiv (c \infty) )$	, ! 17 (( )I: 16)	i
$\forall c ( \omega[c] \Rightarrow ((\ominus c)^D) \equiv (c \infty) )$	! 18 ( $\forall$ I: 1,17)	i
$\square$		
<b>! 30.</b>		i
$\vdash \forall c ( \omega[c] \Rightarrow \mathbf{1} (\ominus c) )$		i
<b>c</b>	, ! 1 (Prem)	i
$\omega[c]$	, ! 2 (Prem)	i
$( \omega[c] \Rightarrow \mathbf{f} (\oplus c) )$	, ! 3 ( $\forall$ E: P10)	i
$\omega[c] \Rightarrow \mathbf{f} (\oplus c)$	, ! 4 (( )E: 3)	i
<b>f</b> ( $\oplus c$ )	, ! 5 ( $\Rightarrow$ E: 2,4)	i
$( \omega[c] \Rightarrow ((\ominus c)^*) \equiv (\oplus c) )$	, ! 6 ( $\forall$ E: P23)	i
$\omega[c] \Rightarrow ((\ominus c)^*) \equiv (\oplus c)$	, ! 7 (( )E: 6)	i
$((\ominus c)^*) \equiv (\oplus c)$	, ! 8 ( $\Rightarrow$ E: 2,7)	i
<b>f</b> ( $\oplus c$ ) $\ \& \ ((\ominus c)^*) \equiv (\oplus c)$	, ! 9 ( $\&$ I: 5,8)	i
$( \mathbf{f} (\oplus c) \ \& \ ((\ominus c)^*) \equiv (\oplus c) \Rightarrow \mathbf{1} (\ominus c) )$	, ! 10 ( $\forall$ E: III9.16)	i

$f(\Theta c) \ \& \ ((\Theta c)^*) \equiv (\Theta c) \Rightarrow 1(\Theta c)$	,! 11 ((E: 10)	i
$1(\Theta c)$	,! 12 ( $\Rightarrow$ E: 9,11)	i
$\omega[c] \Rightarrow 1(\Theta c)$	,! 13 ( $\Rightarrow$ I: 2,12)	i
$(\omega[c] \Rightarrow 1(\Theta c))$	,! 14 ((I: 13)	i
$\forall c (\omega[c] \Rightarrow 1(\Theta c))$	! 15 ( $\forall$ I: 1,14)	i
$\square$		
<b>! 31.</b>		i
$\vdash \forall G \forall n \forall i (f G \ \& \ \omega[n] \ \& \ \omega[i] \ \& \ (G^D)[i]$		
$\Rightarrow (((\Theta n) \circ G)^{(n+i)} = (G'i))$		i
$G, n, i$	,! 1 (Prem)	i
$f G \ \& \ \omega[n] \ \& \ \omega[i] \ \& \ (G^D)[i]$	,! 2 (Prem)	i
$f G$	,! 3 (&E: 2)	i
$\omega[n] \ \& \ \omega[i]$	,! 4 (&E: 2)	i
$\omega[n]$	,! 5 (&E: 2)	i
$(G^D)[i]$	,! 6 (&E: 2)	i
$(G^D)[i] \Rightarrow \exists y G[i, y]$	,! 7 ( $\forall$ E: III5.3)	i
$(G^D)[i] \Rightarrow \exists y G[i, y]$	,! 8 ((E: 7)	i
$\exists y G[i, y]$	,! 9 ( $\Rightarrow$ E: 6,8)	i
$G[i, y]$	,! 10 ( $\exists$ E: 9)	i
$f G \ \& \ G[i, y]$	,! 11 (&I: 3,10)	i
$(f G \ \& \ G[i, y] \Rightarrow (G'i) = y)$	,! 12 ( $\forall$ E: III8.22)	i
$f G \ \& \ G[i, y] \Rightarrow (G'i) = y$	,! 13 ((E: 12)	i
$(G'i) = y$	,! 14 ( $\Rightarrow$ E: 11,13)	i
$(\omega[n] \Rightarrow f(\Theta n))$	,! 15 ( $\forall$ E: P27)	i
$\omega[n] \Rightarrow f(\Theta n)$	,! 16 ((E: 15)	i
$f(\Theta n)$	,! 17 ( $\Rightarrow$ E: 5,16)	i
$f(\Theta n) \ \& \ f G$	,! 18 (&I: 3,17)	i

$( f (\Theta_n) \ \& \ f \ G \Rightarrow f ((\Theta_n) \circ G) )$  ,! 19 ( $\forall E$ : III10.22) i  
 $f (\Theta_n) \ \& \ f \ G \Rightarrow f ((\Theta_n) \circ G)$  ,! 20 ( $()E$ : 19) i  
 $f ((\Theta_n) \circ G)$  ,! 21 ( $\Rightarrow E$ : 18,20) i  
 $( \omega[n] \ \& \ \omega[i] \Rightarrow ((n+i) - n) = i )$  ,! 22 ( $\forall E$ : C6.36) i  
 $\omega[n] \ \& \ \omega[i] \Rightarrow ((n+i) - n) = i$  ,! 23 ( $()E$ : 22) i  
 $((n+i) - n) = i$  ,! 24 ( $\Rightarrow E$ : 4,23) i  
 $( ((n+i)-n) = i \Rightarrow (\Theta_n)[(n+i),i] )$  ,! 25 ( $\forall E$ : 20;  
 $(n+i)$ : C1.7,4) i  
 $((n+i)-n) = i \Rightarrow (\Theta_n)[(n+i),i]$  ,! 26 ( $()E$ : 25) i  
 $(\Theta_n)[(n+i),i]$  ,! 27 ( $\Rightarrow E$ : 24,26) i  
 $(\Theta_n)[(n+i),i] \ \& \ G[i,y]$  ,! 28 ( $\&I$ : 10,27) i  
 $((\Theta_n)[(n+i),i] \ \& \ G[i,y])$  ,! 29 ( $()I$ : 28) i  
 $\exists z((\Theta_n)[(n+i),z] \ \& \ G[z,y])$  ,! 30 ( $\exists I$ : 29) i  
 $( \exists z((\Theta_n)[(n+i),z] \ \& \ G[z,y]) \Rightarrow ((\Theta_n) \circ G)[(n+i),y] )$   
,! 31 ( $\forall E$ : III10.4;  
 $(n+i)$ : C1.7,4) i  
 $\exists z((\Theta_n)[(n+i),z] \ \& \ G[z,y]) \Rightarrow ((\Theta_n) \circ G)[(n+i),y]$   
,! 32 ( $()E$ : 31) i  
 $((\Theta_n) \circ G)[(n+i),y]$  ,! 33 ( $\Rightarrow E$ : 30,32) i  
 $f ((\Theta_n) \circ G) \ \& \ ((\Theta_n) \circ G)[(n+i),y]$  ,! 34 ( $\&I$ : 21,33) i  
 $( f ((\Theta_n) \circ G) \ \& \ ((\Theta_n) \circ G)[(n+i),y]$   
 $\Rightarrow (((\Theta_n) \circ G) \hat{\ } (n+i)) = y )$   
,! 35 ( $\forall E$ : III8.22;  
 $(n+i)$ : C1.7,4) i  
 $f ((\Theta_n) \circ G) \ \& \ ((\Theta_n) \circ G)[(n+i),y]$   
 $\Rightarrow (((\Theta_n) \circ G) \hat{\ } (n+i)) = y$   
,! 36 ( $()E$ : 35) i  
 $((\Theta_n) \circ G) \hat{\ } (n+i)) = y$  ,! 37 ( $\Rightarrow E$ : 34,36) i  
 $((\Theta_n) \circ G) \hat{\ } (n+i)) = (G \hat{\ } i)$  ,! 38 ( $=E$ : 14,37) i  
 $f \ G \ \& \ \omega[n] \ \& \ \omega[i] \ \& \ (G^D)[i] \Rightarrow (((\Theta_n) \circ G) \hat{\ } (n+i)) = (G \hat{\ } i)$

,! 39 ( $\Rightarrow$ I: 2,38) i

(  $\mathbf{f} \mathbf{G} \ \& \ \omega[\mathbf{n}] \ \& \ \omega[\mathbf{i}] \ \& \ (\mathbf{G}^{\mathbf{D}})[\mathbf{i}] \Rightarrow ((\Theta_{\mathbf{n}}) \circ \mathbf{G})'(\mathbf{n}+\mathbf{i}) = (\mathbf{G}'\mathbf{i})$  )

,! 40 ((I: 39) i

$\forall \mathbf{G} \forall \mathbf{n} \forall \mathbf{i}$  (  $\mathbf{f} \mathbf{G} \ \& \ \omega[\mathbf{n}] \ \& \ \omega[\mathbf{i}] \ \& \ (\mathbf{G}^{\mathbf{D}})[\mathbf{i}]$

$\Rightarrow ((\Theta_{\mathbf{n}}) \circ \mathbf{G})'(\mathbf{n}+\mathbf{i}) = (\mathbf{G}'\mathbf{i})$  )

! 41 ( $\forall$ I: 1,40) i

□

! 32. ( $\otimes \mathbf{c}$ ) represents dyadic multiplication, multiplying by  $\mathbf{c}$ .

i

$\mathbb{D} \ \otimes ; (\otimes \mathbf{c}) ; ; (((\otimes \diamond \mathbf{c}) \lceil \omega) \lfloor \omega)$

i

! 33.

i

$\vdash \forall \mathbf{c} \forall \mathbf{x} \forall \mathbf{y}$  (  $\omega[\mathbf{c}] \ \& \ (\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\mathbf{c} \times \mathbf{x}) = \mathbf{y}$  )

i

$\mathbf{c}, \mathbf{x}, \mathbf{y}$

,! 1 (Prem) i

$((\otimes \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}] \Leftrightarrow \otimes[\mathbf{x}, \mathbf{c}, \mathbf{y}])$

,! 2 ( $\forall$ E: III17.2) i

$(\otimes \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}] \Leftrightarrow \otimes[\mathbf{x}, \mathbf{c}, \mathbf{y}]$

,! 3 ((E: 2) i

(  $((\otimes \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\otimes \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{x}]$  )

,! 4 ( $\forall$ E: III7.2) i

$((\otimes \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\otimes \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{x}]$

,! 5 ((E: 4) i

(  $((((\otimes \diamond \mathbf{c}) \lceil \omega) \lfloor \omega)[\mathbf{x}, \mathbf{y}] \Leftrightarrow ((\otimes \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{y}]$  )

,! 6 ( $\forall$ E: III7.50) i

$((((\otimes \diamond \mathbf{c}) \lceil \omega) \lfloor \omega)[\mathbf{x}, \mathbf{y}] \Leftrightarrow ((\otimes \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{y}]$

,! 7 ((E: 6) i

$(\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}] \Leftrightarrow ((\otimes \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{y}]$  ,! 8 ( $\mathbb{D}$ I: P32,7) i

$\omega[\mathbf{c}] \ \& \ (\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}]$

,! 9 (Prem) i

$\omega[\mathbf{c}]$

,! 10 ( $\&$ E: 9) i

$(\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}]$

,! 11 ( $\&$ E: 9) i

$(\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}] \Rightarrow ((\otimes \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{y}]$

,! 12 ( $\Leftrightarrow$ E: 8) i

$((\otimes \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{y}]$

,! 13 ( $\Rightarrow$ E: 11,12) i

$((\otimes \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}]$

,! 14 ( $\&$ E: 13) i

$\omega[\mathbf{y}]$

,! 15 ( $\&$ E: 13) i

$\omega[\mathbf{c}] \ \& \ \omega[\mathbf{y}]$	,! 16 (&I: 10,15)	i
$((\otimes \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}] \Rightarrow (\otimes \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{x}]$	,! 17 ( $\Leftrightarrow$ E: 5)	i
$(\otimes \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{x}]$	,! 18 ( $\Rightarrow$ E: 14,17)	i
$(\otimes \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}]$	,! 19 (&E: 18)	i
$\omega[\mathbf{x}]$	,! 20 (&E: 18)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{c}] \ \& \ \omega[\mathbf{y}]$	,! 21 (&I: 16,20)	i
$(\otimes \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}] \Rightarrow \otimes[\mathbf{x}, \mathbf{c}, \mathbf{y}]$	,! 22 ( $\Leftrightarrow$ E: 3)	i
$\otimes[\mathbf{x}, \mathbf{c}, \mathbf{y}]$	,! 23 ( $\Rightarrow$ E: 19,22)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{c}] \ \& \ \omega[\mathbf{y}] \ \& \ \otimes[\mathbf{x}, \mathbf{c}, \mathbf{y}]$	,! 24 (&I: 21,23)	i
$(\omega[\mathbf{x}] \ \& \ \omega[\mathbf{c}] \ \& \ \omega[\mathbf{y}] \ \& \ \otimes[\mathbf{x}, \mathbf{c}, \mathbf{y}] \Rightarrow (\mathbf{x}\chi\mathbf{c}) = \mathbf{y})$	,! 25 ( $\forall$ E: C7.15)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{c}] \ \& \ \omega[\mathbf{y}] \ \& \ \otimes[\mathbf{x}, \mathbf{c}, \mathbf{y}] \Rightarrow (\mathbf{x}\chi\mathbf{c}) = \mathbf{y}$	,! 26 (( )E: 25)	i
$(\mathbf{x}\chi\mathbf{c}) = \mathbf{y}$	,! 27 ( $\Rightarrow$ E: 24,26)	i
$(\mathbf{x}\chi\mathbf{c}) = \mathbf{y} \Rightarrow (\mathbf{c}\chi\mathbf{x}) = \mathbf{y}$	,! 28 ( $\forall$ E: C8.26)	i
$(\mathbf{x}\chi\mathbf{c}) = \mathbf{y} \Rightarrow (\mathbf{c}\chi\mathbf{x}) = \mathbf{y}$	,! 29 (( )E: 28)	i
$(\mathbf{c}\chi\mathbf{x}) = \mathbf{y}$	,! 30 ( $\Rightarrow$ E: 27,29)	i
$\omega[\mathbf{c}] \ \& \ (\otimes\mathbf{c})[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbf{c}\chi\mathbf{x}) = \mathbf{y}$	,! 31 ( $\Rightarrow$ I: 9,30)	i
$(\mathbf{c}\chi\mathbf{x}) = \mathbf{y}$	,! 32 (Prem)	i
$(\mathbf{c}\chi\mathbf{x}) = \mathbf{y} \Rightarrow (\mathbf{x}\chi\mathbf{c}) = \mathbf{y}$	,! 33 ( $\forall$ E: C8.26)	i
$(\mathbf{c}\chi\mathbf{x}) = \mathbf{y} \Rightarrow (\mathbf{x}\chi\mathbf{c}) = \mathbf{y}$	,! 34 (( )E: 33)	i
$(\mathbf{x}\chi\mathbf{c}) = \mathbf{y}$	,! 35 (Prem)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{c}]$	,! 36 ( $\mathbb{T}$ E: C7.9,35)	i
$\omega[\mathbf{x}]$	,! 37 (&E)	i
$\omega[\mathbf{c}]$	,! 38 (&E)	i
$(\omega[(\mathbf{x}\chi\mathbf{c})] \ \& \ \otimes[\mathbf{x}, \mathbf{c}, (\mathbf{x}\chi\mathbf{c})])$	,! 39 ( $\mathbb{T}$ I: C7.9,36)	i
$\omega[(\mathbf{x}\chi\mathbf{c})] \ \& \ \otimes[\mathbf{x}, \mathbf{c}, (\mathbf{x}\chi\mathbf{c})]$	,! 40 (( )E: 39)	i

$\omega[\mathbf{y}] \ \& \ \otimes[\mathbf{x}, \mathbf{c}, \mathbf{y}]$	, ! 41 (=E: 35,40)	i
$\omega[\mathbf{y}]$	, ! 42 (&E: 41)	i
$\otimes[\mathbf{x}, \mathbf{c}, \mathbf{y}]$	, ! 43 (&E: 41)	i
$\otimes[\mathbf{x}, \mathbf{c}, \mathbf{y}] \Rightarrow (\otimes \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}]$	, ! 44 ( $\Leftrightarrow$ E: 3)	i
$(\otimes \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}]$	, ! 45 ( $\Rightarrow$ E: 43,45)	i
$(\otimes \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{x}]$	, ! 46 (&I: 37,45)	i
$(\otimes \diamond \mathbf{c})[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{x}] \Rightarrow ((\otimes \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}]$	, ! 47 ( $\Leftrightarrow$ E: 5)	i
$((\otimes \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}]$	, ! 48 ( $\Rightarrow$ E: 46,47)	i
$((\otimes \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{y}]$	, ! 49 (&I: 42,48)	i
$((\otimes \diamond \mathbf{c}) \lceil \omega)[\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{y}] \Rightarrow (\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}]$	, ! 50 ( $\Leftrightarrow$ E: 8)	i
$(\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}]$	, ! 51 ( $\Rightarrow$ E: 49,50)	i
$\omega[\mathbf{c}] \ \& \ (\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}]$	, ! 52 (&I: 38,51)	i
$(\mathbf{cXx}) = \mathbf{y} \Rightarrow \omega[\mathbf{c}] \ \& \ (\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}]$	, ! 52 ( $\Rightarrow$ I: 32,52)	i
$\omega[\mathbf{c}] \ \& \ (\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\mathbf{cXx}) = \mathbf{y}$	, ! 53 ( $\Leftrightarrow$ I: 31,52)	i
$(\omega[\mathbf{c}] \ \& \ (\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\mathbf{cXx}) = \mathbf{y})$	, ! 54 (( )I: 53)	i
$\forall \mathbf{c} \forall \mathbf{x} \forall \mathbf{y} ( \omega[\mathbf{c}] \ \& \ (\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\mathbf{cXx}) = \mathbf{y} )$	! 55 ( $\forall$ I: 1,54)	i

□

! 34.

$\vdash \forall \mathbf{c} \forall \mathbf{x} \forall \mathbf{y} ( \omega[\mathbf{c}] \ \& \ (\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbf{cXx}) = \mathbf{y} )$		i
$\mathbf{c}, \mathbf{x}, \mathbf{y}$	, ! 1 (Prem)	i
$( \omega[\mathbf{c}] \ \& \ (\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\mathbf{cXx}) = \mathbf{y} )$	, ! 2 ( $\forall$ E: P33)	i
$\omega[\mathbf{c}] \ \& \ (\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\mathbf{cXx}) = \mathbf{y}$	, ! 3 (( )E: 2)	i
$\omega[\mathbf{c}] \ \& \ (\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbf{cXx}) = \mathbf{y}$	, ! 4 ( $\Leftrightarrow$ E: 3)	i
$( \omega[\mathbf{c}] \ \& \ (\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbf{cXx}) = \mathbf{y} )$	, ! 5 (( )I: 4)	i
$\forall \mathbf{c} \forall \mathbf{x} \forall \mathbf{y} ( \omega[\mathbf{c}] \ \& \ (\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbf{cXx}) = \mathbf{y} )$	! 6 ( $\forall$ I: 1,5)	i

□

! 35.

$\vdash \forall c \forall x \forall y ( (cXx) = y \Rightarrow (\otimes c)[x, y] )$		i
$c, x, y$	, ! 1 (Prem)	i
$(cXx) = y$	, ! 2 (Prem)	i
$( \omega[c] \ \& \ (\otimes c)[x, y] \Leftrightarrow (cXx) = y )$	, ! 3 ( $\forall E$ : P33)	i
$\omega[c] \ \& \ (\otimes c)[x, y] \Leftrightarrow (cXx) = y$	, ! 4 ( $(\ )E$ : 3)	i
$(cXx) = y \Rightarrow \omega[c] \ \& \ (\otimes c)[x, y]$	, ! 5 ( $\Leftrightarrow E$ : 4)	i
$\omega[c] \ \& \ (\otimes c)[x, y]$	, ! 6 ( $\Rightarrow E$ : 2, 5)	i
$(\otimes c)[x, y]$	, ! 7 ( $\& E$ : 6)	i
$(cXx) = y \Rightarrow (\otimes c)[x, y]$	, ! 8 ( $\Rightarrow I$ : 2, 7)	i
$( (cXx) = y \Rightarrow (\otimes c)[x, y] )$	, ! 9 ( $(\ )I$ : 8)	i
$\forall c \forall x \forall y ( (cXx) = y \Rightarrow (\otimes c)[x, y] )$	! 10 ( $\forall I$ : 1, 9)	i

□

! 36.

$\vdash \forall c \forall x ( \omega[c] \ \& \ \omega[x] \Rightarrow (\otimes c)[x, (cXx)] )$		i
$c, x$	, ! 1 (Prem)	i
$\omega[c] \ \& \ \omega[x]$	, ! 2 (Prem)	i
$(cXx) = (cXx)$	, ! 3 ( $=I$ ; $(cXx)$ : C7.9, 2)	i
$( (cXx) = (cXx) \Rightarrow (\otimes c)[x, (cXx)] )$	, ! 4 ( $\forall E$ : P35; $(cXx)$ : C7.9, 2)	i
$(cXx) = (cXx) \Rightarrow (\otimes c)[x, (cXx)]$	, ! 5 ( $(\ )E$ : 4)	i
$(\otimes c)[x, (cXx)]$	, ! 6 ( $\Rightarrow E$ : 3, 5)	i
$\omega[c] \ \& \ \omega[x] \Rightarrow (\otimes c)[x, (cXx)]$	, ! 7 ( $\Rightarrow I$ : 2, 6)	i
$( \omega[c] \ \& \ \omega[x] \Rightarrow (\otimes c)[x, (cXx)] )$	, ! 8 ( $(\ )I$ : 7)	i
$\forall c \forall x ( \omega[c] \ \& \ \omega[x] \Rightarrow (\otimes c)[x, (cXx)] )$	! 9 ( $\forall I$ : 1, 8)	i

□

! 37.

$\vdash \forall c \forall x \forall y ( \omega[c] \ \& \ (\otimes c)[x, y] \Rightarrow \omega[x] \ \& \ \omega[y] )$		i
$c, x, y$	, ! 1 (Prem)	i

$\omega[\mathbf{c}] \ \& \ (\otimes\mathbf{c})[\mathbf{x},\mathbf{y}]$	,! 2 (Prem)	i
$( \ \omega[\mathbf{c}] \ \& \ (\otimes\mathbf{c})[\mathbf{x},\mathbf{y}] \Rightarrow (\mathbf{c}\chi\mathbf{x}) = \mathbf{y} \ )$	,! 3 ( $\forall\mathbf{E}$ : P34)	i
$\omega[\mathbf{c}] \ \& \ (\otimes\mathbf{c})[\mathbf{x},\mathbf{y}] \Leftrightarrow (\mathbf{c}\chi\mathbf{x}) = \mathbf{y}$	,! 4 ( $(\ )\mathbf{E}$ : 3)	i
$(\mathbf{c}\chi\mathbf{x}) = \mathbf{y}$	,! 5 ( $\Rightarrow\mathbf{E}$ : 2,4)	i
$\omega[\mathbf{c}] \ \& \ \omega[\mathbf{x}]$	,! 6 ( $\mathbb{T}\mathbf{E}$ : C7.9,5)	i
$\omega[\mathbf{x}]$	,! 7 ( $\&\mathbf{E}$ : 6)	i
$( \ \omega[\mathbf{c}] \ \& \ \omega[\mathbf{x}] \Rightarrow \omega[(\mathbf{c}\chi\mathbf{x})] \ )$	,! 8 ( $\forall\mathbf{E}$ : C1.8)	i
$\omega[\mathbf{c}] \ \& \ \omega[\mathbf{x}] \Rightarrow \omega[(\mathbf{c}\chi\mathbf{x})]$	,! 9 ( $(\ )\mathbf{E}$ : 8)	i
$\omega[(\mathbf{c}\chi\mathbf{x})]$	,! 10 ( $\Rightarrow\mathbf{E}$ : 6,9)	i
$\omega[\mathbf{y}]$	,! 11 ( $=\mathbf{E}$ : 5,10)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}]$	,! 12 ( $\&\mathbf{I}$ : 7,11)	i
$\omega[\mathbf{c}] \ \& \ (\otimes\mathbf{c})[\mathbf{x},\mathbf{y}] \Rightarrow \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}]$	,! 13 ( $\Rightarrow\mathbf{I}$ : 2,12)	i
$( \ \omega[\mathbf{c}] \ \& \ (\otimes\mathbf{c})[\mathbf{x},\mathbf{y}] \Rightarrow \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ )$	,! 14 ( $(\ )\mathbf{E}$ : 13)	i
$\forall\mathbf{c}\forall\mathbf{x}\forall\mathbf{y} ( \ \omega[\mathbf{c}] \ \& \ (\otimes\mathbf{c})[\mathbf{x},\mathbf{y}] \Rightarrow \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ )$	! 15 ( $\forall\mathbf{I}$ : 1,14)	i

□

! 38. Multiplying by 1 is the identity on  $\omega$ . i

$\vdash (\otimes 1) \equiv (\mathbb{I}\omega)$		i
$\mathbf{x},\mathbf{y}$	,! 1 (Prem)	i
$( \ \omega[1] \ \& \ (\otimes 1)[\mathbf{x},\mathbf{y}] \Leftrightarrow (1\chi\mathbf{x}) = \mathbf{y} \ )$	,! 2 ( $\forall\mathbf{E}$ : P33)	i
$\omega[1] \ \& \ (\otimes 1)[\mathbf{x},\mathbf{y}] \Leftrightarrow (1\chi\mathbf{x}) = \mathbf{y}$	,! 3 ( $(\ )\mathbf{E}$ : 2)	i
$( \ \omega[\mathbf{x}] \Rightarrow (1\chi\mathbf{x}) = \mathbf{x} \ )$	,! 4 ( $\forall\mathbf{E}$ : C8.14)	i
$\omega[\mathbf{x}] \Rightarrow (1\chi\mathbf{x}) = \mathbf{x}$	,! 5 ( $(\ )\mathbf{E}$ : 4)	i
$( \ (\mathbb{I}\omega)[\mathbf{x},\mathbf{y}] \Leftrightarrow \mathbf{x} = \mathbf{y} \ \& \ \omega[\mathbf{x}] \ )$	,! 6 ( $\forall\mathbf{E}$ : III11.17)	i
$(\mathbb{I}\omega)[\mathbf{x},\mathbf{y}] \Leftrightarrow \mathbf{x} = \mathbf{y} \ \& \ \omega[\mathbf{x}]$	,! 7 ( $(\ )\mathbf{E}$ : 6)	i
$(\otimes 1)[\mathbf{x},\mathbf{y}]$	,! 8 (Prem)	i
$\omega[1] \ \& \ (\otimes 1)[\mathbf{x},\mathbf{y}]$	,! 9 ( $\&\mathbf{I}$ : IV9.2,8)	i

$\omega[1] \ \& \ (\otimes 1)[\mathbf{x}, \mathbf{y}] \Rightarrow (1\mathbf{x}\mathbf{x}) = \mathbf{y}$	, ! 10 ( $\Leftrightarrow$ E: 3)	i
$(1\mathbf{x}\mathbf{x}) = \mathbf{y}$	, ! 11 ( $\Rightarrow$ E: 9,10)	i
$\omega[1] \ \& \ \omega[\mathbf{x}]$	, ! 12 ( $\mathbb{T}$ E: C7.9,11)	i
$\omega[\mathbf{x}]$	, ! 13 ( $\&$ E: 12)	i
$(1\mathbf{x}\mathbf{x}) = \mathbf{x}$	, ! 14 ( $\Rightarrow$ E: 5,13)	i
$\mathbf{x} = \mathbf{y}$	, ! 15 ( $=$ E: 11,14)	i
$\mathbf{x} = \mathbf{y} \ \& \ \omega[\mathbf{x}]$	, ! 16 ( $\&$ I: 13,15)	i
$\mathbf{x} = \mathbf{y} \ \& \ \omega[\mathbf{x}] \Rightarrow (\mathbb{I}\omega)[\mathbf{x}, \mathbf{y}]$	, ! 17 ( $\Leftrightarrow$ E: 7)	i
$(\mathbb{I}\omega)[\mathbf{x}, \mathbf{y}]$	, ! 18 ( $\Rightarrow$ E: 16,17)	i
$(\otimes 1)[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbb{I}\omega)[\mathbf{x}, \mathbf{y}]$	, ! 19 ( $\Rightarrow$ I: 8,18)	i
$(\mathbb{I}\omega)[\mathbf{x}, \mathbf{y}]$	, ! 20 (Prem)	i
$(\mathbb{I}\omega)[\mathbf{x}, \mathbf{y}] \Rightarrow \mathbf{x} = \mathbf{y} \ \& \ \omega[\mathbf{x}]$	, ! 21 ( $\Leftrightarrow$ E: 7)	i
$\mathbf{x} = \mathbf{y} \ \& \ \omega[\mathbf{x}]$	, ! 22 ( $\Rightarrow$ E: 20,21)	i
$\mathbf{x} = \mathbf{y}$	, ! 23 ( $\&$ E: 22)	i
$\omega[\mathbf{x}]$	, ! 24 ( $\&$ E: 22)	i
$(1\mathbf{x}\mathbf{x}) = \mathbf{x}$	, ! 25 ( $\Rightarrow$ E: 5,24)	i
$(1\mathbf{x}\mathbf{x}) = \mathbf{y}$	, ! 26 ( $=$ E: 23,25)	i
$(1\mathbf{x}\mathbf{x}) = \mathbf{y} \Rightarrow \omega[1] \ \& \ (\otimes 1)[\mathbf{x}, \mathbf{y}]$	, ! 27 ( $\Leftrightarrow$ E: 3)	i
$\omega[1] \ \& \ (\otimes 1)[\mathbf{x}, \mathbf{y}]$	, ! 28 ( $\Rightarrow$ E: 26,27)	i
$(\otimes 1)[\mathbf{x}, \mathbf{y}]$	, ! 29 ( $\&$ E: 28)	i
$(\mathbb{I}\omega)[\mathbf{x}, \mathbf{y}] \Rightarrow (\otimes 1)[\mathbf{x}, \mathbf{y}]$	, ! 30 ( $\Rightarrow$ I: 20,29)	i
$(\otimes 1)[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\mathbb{I}\omega)[\mathbf{x}, \mathbf{y}]$	, ! 31 ( $\Leftrightarrow$ I: 19,30)	i
$( (\otimes 1)[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\mathbb{I}\omega)[\mathbf{x}, \mathbf{y}] )$	, ! 32 ( $(())$ I: 31)	i
$\forall \mathbf{x} \forall \mathbf{y} ( (\otimes 1)[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\mathbb{I}\omega)[\mathbf{x}, \mathbf{y}] )$	, ! 33 ( $\forall$ I: 1,32)	i
$(\otimes 1) \equiv (\mathbb{I}\omega)$	! 34 ( $\mathbb{S}$ I: III1.5)	i
$\square$		
! 39.		i
$\vdash \forall c ( \omega[c] \Rightarrow ((\otimes c)^D) \equiv \omega )$		i

<b>c</b>	,! 1 (Prem)	i
$\omega[\mathbf{c}]$	,! 2 (Prem)	i
$(\forall x(\exists y (\otimes \mathbf{c})[x,y] \Leftrightarrow \omega[x]) \Rightarrow ((\otimes \mathbf{c})^D) \equiv \omega)$	,! 3 ( $\forall E$ : III5.13)	i
$\forall x(\exists y (\otimes \mathbf{c})[x,y] \Leftrightarrow \omega[x]) \Rightarrow ((\otimes \mathbf{c})^D) \equiv \omega$	,! 4 ( $()E$ : 3)	i
<b>x</b>	,! 5 (Prem)	i
$\exists y (\otimes \mathbf{c})[\mathbf{x},y]$	,! 6 (Prem)	i
$(\otimes \mathbf{c})[\mathbf{x},y]$	,! 7 ( $\exists E$ : 6)	i
$\omega[\mathbf{c}] \ \& \ (\otimes \mathbf{c})[\mathbf{x},y]$	,! 8 ( $\&I$ : 2,7)	i
$(\omega[\mathbf{c}] \ \& \ (\otimes \mathbf{c})[\mathbf{x},y] \Rightarrow (\mathbf{cXx}) = \mathbf{y})$	,! 9 ( $\forall E$ : P34)	i
$\omega[\mathbf{c}] \ \& \ (\otimes \mathbf{c})[\mathbf{x},y] \Rightarrow (\mathbf{cXx}) = \mathbf{y}$	,! 10 ( $()E$ : 9)	i
$(\mathbf{cXx}) = \mathbf{y}$	,! 11 ( $\Rightarrow E$ : 8,10)	i
$\omega[\mathbf{c}] \ \& \ \omega[\mathbf{x}]$	,! 12 ( $\mathbb{T}E$ : C7.9,11)	i
$\omega[\mathbf{x}]$	,! 13 ( $\&E$ : 12)	i
$\exists y (\otimes \mathbf{c})[\mathbf{x},y] \Rightarrow \omega[\mathbf{x}]$	,! 14 ( $\Rightarrow I$ : 6,13)	i
$\omega[\mathbf{x}]$	,! 15 (Prem)	i
$\omega[\mathbf{c}] \ \& \ \omega[\mathbf{x}]$	,! 16 ( $\&I$ : 2,15)	i
$(\omega[\mathbf{c}] \ \& \ \omega[\mathbf{x}] \Rightarrow (\otimes \mathbf{c})[\mathbf{x},(\mathbf{cXx})])$	,! 17 ( $\forall E$ : P36)	i
$\omega[\mathbf{c}] \ \& \ \omega[\mathbf{x}] \Rightarrow (\otimes \mathbf{c})[\mathbf{x},(\mathbf{cXx})]$	,! 18 ( $()E$ : 17)	i
$(\otimes \mathbf{c})[\mathbf{x},(\mathbf{cXx})]$	,! 19 ( $\Rightarrow E$ : 16,18)	i
$\exists y (\otimes \mathbf{c})[\mathbf{x},y]$	,! 20 ( $\exists I$ : 19; $(\mathbf{cXx})$ : C7.9,16)	i
$\omega[\mathbf{x}] \Rightarrow \exists y (\otimes \mathbf{c})[\mathbf{x},y]$	,! 21 ( $\Rightarrow I$ : 15,20)	i
$\exists y (\otimes \mathbf{c})[\mathbf{x},y] \Leftrightarrow \omega[\mathbf{x}]$	,! 22 ( $\Leftrightarrow I$ : 14,21)	i
$(\exists y (\otimes \mathbf{c})[\mathbf{x},y] \Leftrightarrow \omega[\mathbf{x}])$	,! 23 ( $()I$ : 22)	i
$\forall x(\exists y (\otimes \mathbf{c})[x,y] \Leftrightarrow \omega[x])$	,! 24 ( $\forall I$ : 5,23)	i
$((\otimes \mathbf{c})^D) \equiv \omega$	,! 25 ( $\Rightarrow E$ : 4,24)	i

$\omega[\mathbf{c}] \Rightarrow ((\otimes \mathbf{c})^D) \equiv \omega$	,!	26 ( $\Rightarrow$ I: 2,25)	i
$(\omega[\mathbf{c}] \Rightarrow ((\otimes \mathbf{c})^D) \equiv \omega)$	,!	27 ((I: 26)	i
$\forall \mathbf{c} (\omega[\mathbf{c}] \Rightarrow ((\otimes \mathbf{c})^D) \equiv \omega)$	!	28 ( $\forall$ I: 1,27)	i
$\square$			
<b>! 40.</b>			i
$\vdash \forall \mathbf{c} (\omega[\mathbf{c}] \Rightarrow \mathbf{f}(\otimes \mathbf{c}))$			i
$\mathbf{c}$	,!	1 (Prem)	i
$\omega[\mathbf{c}]$	,!	2 (Prem)	i
$\mathbf{x}, \mathbf{y}, \mathbf{z}$	,!	3 (Prem)	i
$(\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}] \ \& \ (\otimes \mathbf{c})[\mathbf{x}, \mathbf{z}]$	,!	4 (Prem)	i
$(\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}]$	,!	5 ( $\&$ E: 4)	i
$(\otimes \mathbf{c})[\mathbf{x}, \mathbf{z}]$	,!	6 ( $\&$ E: 4)	i
$\omega[\mathbf{c}] \ \& \ (\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}]$	,!	7 ( $\&$ I: 2,5)	i
$(\omega[\mathbf{c}] \ \& \ (\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbf{cXx}) = \mathbf{y})$	,!	8 ( $\forall$ E: P34)	i
$\omega[\mathbf{c}] \ \& \ (\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbf{cXx}) = \mathbf{y}$	,!	9 ((E: 8)	i
$(\mathbf{cXx}) = \mathbf{y}$	,!	10 ( $\Rightarrow$ E: 7,9)	i
$\omega[\mathbf{c}] \ \& \ (\otimes \mathbf{c})[\mathbf{x}, \mathbf{z}]$	,!	11 ( $\&$ I: 2,6)	i
$(\omega[\mathbf{c}] \ \& \ (\otimes \mathbf{c})[\mathbf{x}, \mathbf{z}] \Rightarrow (\mathbf{cXx}) = \mathbf{z})$	,!	12 ( $\forall$ E: P34)	i
$\omega[\mathbf{c}] \ \& \ (\otimes \mathbf{c})[\mathbf{x}, \mathbf{z}] \Rightarrow (\mathbf{cXx}) = \mathbf{z}$	,!	13 ((E: 12)	i
$(\mathbf{cXx}) = \mathbf{z}$	,!	14 ( $\Rightarrow$ E: 11,13)	i
$\mathbf{y} = \mathbf{z}$	,!	15 (=E: 10,14)	i
$(\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}] \ \& \ (\otimes \mathbf{c})[\mathbf{x}, \mathbf{z}] \Rightarrow \mathbf{y} = \mathbf{z}$	,!	16 ( $\Rightarrow$ I: 4,15)	i
$((\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}] \ \& \ (\otimes \mathbf{c})[\mathbf{x}, \mathbf{z}] \Rightarrow \mathbf{y} = \mathbf{z})$	,!	17 ((I: 16)	i
$\forall \mathbf{x} \forall \mathbf{y} \forall \mathbf{z} ((\otimes \mathbf{c})[\mathbf{x}, \mathbf{y}] \ \& \ (\otimes \mathbf{c})[\mathbf{x}, \mathbf{z}] \Rightarrow \mathbf{y} = \mathbf{z})$	,!	18 ( $\forall$ I: 3,17)	i
$\mathbf{f}(\otimes \mathbf{c})$	,!	19 ( $\S$ I: III8.1,18)	i
$\omega[\mathbf{c}] \Rightarrow \mathbf{f}(\otimes \mathbf{c})$	,!	20 ( $\Rightarrow$ I: 2,19)	i
$(\omega[\mathbf{c}] \Rightarrow \mathbf{f}(\otimes \mathbf{c}))$	,!	21 ((I: 20)	i

$\forall c ( \omega[c] \Rightarrow f ( \otimes c ) )$  ! 22 ( $\forall I$ : 1,21) i

□

! 41. i

$\vdash \forall c ( \omega[c] \Rightarrow ( \otimes c ) \mathbb{F} \omega )$  i

$c$  ,! 1 (Prem) i

$\omega[c]$  ,! 2 (Prem) i

$( \omega[c] \Rightarrow ( ( \otimes c )^D ) \equiv \omega )$  ,! 3 ( $\forall E$ : P39) i

$\omega[c] \Rightarrow ( ( \otimes c )^D ) \equiv \omega$  ,! 4 ( $( )E$ : 3) i

$( ( \otimes c )^D ) \equiv \omega$  ,! 5 ( $\Rightarrow E$ : 2,4) i

$( \omega[c] \Rightarrow f ( \otimes c ) )$  ,! 6 ( $\forall E$ : P40) i

$\omega[c] \Rightarrow f ( \otimes c )$  ,! 7 ( $( )E$ : 6) i

$f ( \otimes c )$  ,! 8 ( $\Rightarrow E$ : 2,7) i

$( ( \otimes c )^D ) \equiv \omega \ \& \ f ( \otimes c )$  ,! 9 ( $\&I$ : 5,8) i

$( \otimes c ) \mathbb{F} \omega$  ,! 10 ( $\$I$ : III8.10,9) i

$\omega[c] \Rightarrow ( \otimes c ) \mathbb{F} \omega$  ,! 11 ( $\Rightarrow I$ : 2,10) i

$( \omega[c] \Rightarrow ( \otimes c ) \mathbb{F} \omega )$  ,! 12 ( $( )I$ : 11) i

$\forall c ( \omega[c] \Rightarrow ( \otimes c ) \mathbb{F} \omega )$  ! 13 ( $\forall I$ : 1,12) i

□

! 42. i

$\vdash \forall c \forall x ( \omega[c] \ \& \ \omega[x] \Rightarrow ( ( \otimes c ) \hat{\ } x ) = ( cXx ) )$  i

$c, x$  ,! 1 (Prem) i

$\omega[c] \ \& \ \omega[x]$  ,! 2 (Prem) i

$\omega[c]$  ,! 3 ( $\&E$ : 2) i

$( \omega[c] \Rightarrow f ( \otimes c ) )$  ,! 4 ( $\forall E$ : P40) i

$\omega[c] \Rightarrow f ( \otimes c )$  ,! 5 ( $( )E$ : 4) i

$f ( \otimes c )$  ,! 6 ( $\Rightarrow E$ : 3,5) i

$( \omega[c] \ \& \ \omega[x] \Rightarrow ( \otimes c ) [ x , ( cXx ) ] )$  ,! 7 ( $\forall E$ : P36) i

$\omega[c] \ \& \ \omega[x] \Rightarrow (\otimes c)[x, (cXx)]$	, ! 8 ((E: 7)	i
$(\otimes c)[x, (cXx)]$	, ! 9 ( $\Rightarrow$ E: 2, 8)	i
<b>f</b> $(\otimes c) \ \& \ (\otimes c)[x, (cXx)]$	, ! 10 (&I: 6, 9)	i
$( \mathbf{f} (\otimes c) \ \& \ (\otimes c)[x, (cXx)] \Rightarrow ((\otimes c)'x) = (cXx) )$	, ! 11 ( $\forall$ E: III8.22; (cXx): C7.9, 2)	i
<b>f</b> $(\otimes c) \ \& \ (\otimes c)[x, (cXx)] \Rightarrow ((\otimes c)'x) = (cXx)$	, ! 12 ((E: 11)	i
$((\otimes c)'x) = (cXx)$	, ! 13 ( $\Rightarrow$ E: 10, 12)	i
$\omega[c] \ \& \ \omega[x] \Rightarrow ((\otimes c)'x) = (cXx)$	, ! 14 ( $\Rightarrow$ I: 2, 13)	i
$( \omega[c] \ \& \ \omega[x] \Rightarrow ((\otimes c)'x) = (cXx) )$	, ! 15 ((I: 14)	i
$\forall c \forall x ( \omega[c] \ \& \ \omega[x] \Rightarrow ((\otimes c)'x) = (cXx) )$	! 16 ( $\forall$ I: 1, 15)	i
$\square$		

! 43.

$\vdash \forall R \forall c \forall x ( \omega[c] \ \& \ \omega[(R'x)] \Rightarrow ((R \circ (\otimes c))'x) = (cX(R'x)) )$	i	
<b>R, c, x</b>	, ! 1 (Prem)	i
$\omega[c] \ \& \ \omega[(R'x)]$	, ! 2 (Prem)	i
$\omega[c]$	, ! 3 (&E: 2)	i
$\omega[(R'x)]$	, ! 4 (&E: 2)	i
$( \omega[c] \Rightarrow \mathbf{f} (\otimes c) )$	, ! 5 ( $\forall$ E: P40)	i
$\omega[c] \Rightarrow \mathbf{f} (\otimes c)$	, ! 6 ((E: 5)	i
<b>f</b> $(\otimes c)$	, ! 7 ( $\Rightarrow$ E: 3, 6)	i
$( \omega[c] \Rightarrow ((\otimes c)^D) \equiv \omega )$	, ! 8 ( $\forall$ E: P39)	i
$\omega[c] \Rightarrow ((\otimes c)^D) \equiv \omega$	, ! 9 ((E: 8)	i
$((\otimes c)^D) \equiv \omega$	, ! 10 ( $\Rightarrow$ E: 3, 9)	i
$\omega[(R'x)] \ \& \ ((\otimes c)^D) \equiv \omega$	, ! 11 (&I: 4, 10)	i
<b>f</b> <b>R</b> $\ \& \ (R^D)[x]$	, ! 12 ( $\mathbb{T}$ E: III8.20, 4)	i
$( \omega[(R'x)] \ \& \ ((\otimes c)^D) \equiv \omega \Rightarrow ((\otimes c)^D)[(R'x)] )$	, ! 13 ( $\forall$ E: III.36;	

