

! CHAPTER 7

RESTRICTIONS ;

! In this chapter the concept of restrictions is introduced, both for domains and images. The (domain) restriction of **R** to **A**, which has greater importance, is written $(\mathbf{R} \upharpoonright \mathbf{A})$. The image restriction of **R**, of less significance and accordingly much less developed, is written $(\mathbf{R} \downharpoonright \mathbf{B})$. **x** bears $(\mathbf{R} \upharpoonright \mathbf{A})$ to **y** if and only if both **x** satisfies **A** and **x** bears **R** to **y**, and **x** bears $(\mathbf{R} \downharpoonright \mathbf{B})$ to **y** if and only if both **y** satisfies **B** and **x** bears **R** to **y**.

There is an evident duality between the two concepts, which will be established in P51 and P52. ;

! The organization is as follows:

- P1-P48: domain restrictions
- P1: definition
- P2-P7: fundamental proposition and corollaries
- P8-P13: concern the maintenance of inclusion and equivalence by restrictions
- P14-P16: have terms containing two restrictions
- P17-P19: concern (two-place) unions
- P20-P22: concern the (two-place) empty predicate
- P23-P37: involve domains
- P38-P48: involve images
- P49-P53: image restrictions ;

! 1. \upharpoonright represents restriction. ;

$\mathbb{D} \upharpoonright ; (\mathbf{R} \upharpoonright \mathbf{A}) ; ; \{a,b : \mathbf{R}[a,b] \ \& \ \mathbf{A}[a]\}$;

! 2. **Fundamental Proposition of Restrictions.** ;

$\vdash \forall \mathbf{R} \forall \mathbf{A} \forall \mathbf{x} \forall \mathbf{y} ((\mathbf{R} \upharpoonright \mathbf{A})[\mathbf{x},\mathbf{y}] \Leftrightarrow \mathbf{R}[\mathbf{x},\mathbf{y}] \ \& \ \mathbf{A}[\mathbf{x}])$;

R, A ,! 1 (Prem) ;

$\forall \mathbf{x} \forall \mathbf{y} (\{a,b : \mathbf{R}[a,b] \ \& \ \mathbf{A}[a]\}[\mathbf{x},\mathbf{y}] \Leftrightarrow \mathbf{R}[\mathbf{x},\mathbf{y}] \ \& \ \mathbf{A}[\mathbf{x}])$,! 2 (Pred) ;

$\forall \mathbf{x} \forall \mathbf{y} ((\mathbf{R} \upharpoonright \mathbf{A})[\mathbf{x},\mathbf{y}] \Leftrightarrow \mathbf{R}[\mathbf{x},\mathbf{y}] \ \& \ \mathbf{A}[\mathbf{x}])$,! 3 ($\mathbb{D}\mathbb{I}$: P1,2) ;

$\forall \mathbf{R} \forall \mathbf{A} \forall \mathbf{x} \forall \mathbf{y} ((\mathbf{R} \upharpoonright \mathbf{A})[\mathbf{x},\mathbf{y}] \Leftrightarrow \mathbf{R}[\mathbf{x},\mathbf{y}] \ \& \ \mathbf{A}[\mathbf{x}])$;

! 4 ($\forall\mathbb{I}$: 1,3) ;

□

! 3. **Fundamental Proposition of Restrictions, First Half.** ;

$\vdash \forall \mathbf{R} \forall \mathbf{A} \forall \mathbf{x} \forall \mathbf{y} ((\mathbf{R} \upharpoonright \mathbf{A})[\mathbf{x},\mathbf{y}] \Rightarrow \mathbf{R}[\mathbf{x},\mathbf{y}] \ \& \ \mathbf{A}[\mathbf{x}])$;

R, A, x, y ,! 1 (Prem) ;

$((\mathbf{R} \upharpoonright \mathbf{A})[\mathbf{x},\mathbf{y}] \Leftrightarrow \mathbf{R}[\mathbf{x},\mathbf{y}] \ \& \ \mathbf{A}[\mathbf{x}])$,! 2 ($\forall\mathbb{E}$: P2) ;

$(\mathbf{R} \upharpoonright \mathbf{A})[\mathbf{x},\mathbf{y}] \Leftrightarrow \mathbf{R}[\mathbf{x},\mathbf{y}] \ \& \ \mathbf{A}[\mathbf{x}]$,! 3 ((\mathbb{E}) : 2) ;

$(R \ulcorner A)[x,y] \Rightarrow R[x,y] \ \& \ A[x]$,! 4 ($\Leftrightarrow E$: 3) i
 $((R \ulcorner A)[x,y] \Rightarrow R[x,y] \ \& \ A[x])$,! 5 ($(\)I$: 4) i
 $\forall R \forall A \forall x \forall y ((R \ulcorner A)[x,y] \Rightarrow R[x,y] \ \& \ A[x])$! 6 ($\forall I$: 1,5) i
 \square

! 4. Fundamental Proposition of Restrictions, Second Half.

$\vdash \forall R \forall A \forall x \forall y (R[x,y] \ \& \ A[x] \Rightarrow (R \ulcorner A)[x,y])$ i
 R, A, x, y ,! 1 (Prem) i
 $((R \ulcorner A)[x,y] \Leftrightarrow R[x,y] \ \& \ A[x])$,! 2 ($\forall E$: P2) i
 $(R \ulcorner A)[x,y] \Leftrightarrow R[x,y] \ \& \ A[x]$,! 3 ($(\)E$: 2) i
 $R[x,y] \ \& \ A[x] \Rightarrow (R \ulcorner A)[x,y]$,! 4 ($\Leftrightarrow E$: 3) i
 $(R[x,y] \ \& \ A[x] \Rightarrow (R \ulcorner A)[x,y])$,! 5 ($(\)I$: 4) i
 $\forall R \forall A \forall x \forall y (R[x,y] \ \& \ A[x] \Rightarrow (R \ulcorner A)[x,y])$! 6 ($\forall I$: 1,5) i
 \square

! 5.

$\vdash \forall R \forall A \forall x \forall y ((R \ulcorner A)[x,y] \Rightarrow R[x,y])$ i
 R, A, x, y ,! 1 (Prem) i
 $(R \ulcorner A)[x,y]$,! 2 (Prem) i
 $((R \ulcorner A)[x,y] \Rightarrow R[x,y] \ \& \ A[x])$,! 3 ($\forall E$: P3) i
 $(R \ulcorner A)[x,y] \Rightarrow R[x,y] \ \& \ A[x]$,! 4 ($(\)E$: 3) i
 $R[x,y] \ \& \ A[x]$,! 5 ($\Rightarrow E$: 2,4) i
 $R[x,y]$,! 6 ($\&E$: 5) i
 $(R \ulcorner A)[x,y] \Rightarrow R[x,y]$,! 7 ($\Rightarrow I$: 2,6) i
 $((R \ulcorner A)[x,y] \Rightarrow R[x,y])$,! 8 ($(\)I$: 7) i
 $\forall R \forall A \forall x \forall y ((R \ulcorner A)[x,y] \Rightarrow R[x,y])$! 9 ($\forall I$: 1,8) i
 \square

! 6.

$\vdash \forall R \forall A \forall x \forall y ((R \ulcorner A)[x,y] \Rightarrow A[x])$ i
 R, A, x, y ,! 1 (Prem) i

$(R \upharpoonright A)[x, y]$, ! 2 (Prem)	i
$((R \upharpoonright A)[x, y] \Rightarrow R[x, y] \ \& \ A[x])$, ! 3 ($\forall E$: P3)	i
$(R \upharpoonright A)[x, y] \Rightarrow R[x, y] \ \& \ A[x]$, ! 4 ($(\Rightarrow)E$: 3)	i
$R[x, y] \ \& \ A[x]$, ! 5 ($\Rightarrow E$: 2, 4)	i
$A[x]$, ! 6 ($\&E$: 5)	i
$(R \upharpoonright A)[x, y] \Rightarrow A[x]$, ! 7 ($\Rightarrow I$: 2, 6)	i
$((R \upharpoonright A)[x, y] \Rightarrow A[x])$, ! 8 ($(\Rightarrow)I$: 7)	i
$\forall R \forall A \forall x \forall y ((R \upharpoonright A)[x, y] \Rightarrow A[x])$! 9 ($\forall I$: 1, 8)	i

□

! 7.

$\vdash \forall R \forall A (R \upharpoonright A) \subseteq R$		i
R, A	, ! 1 (Prem)	i
$\forall x \forall y ((R \upharpoonright A)[x, y] \Rightarrow R[x, y])$, ! 2 ($\forall E$: P5)	i
$(R \upharpoonright A) \subseteq R$, ! 3 ($\I: C1.1, 2)	i
$\forall R \forall A (R \upharpoonright A) \subseteq R$! 4 ($\forall I$: 1, 3)	i

□

! 8. Restrictions maintain inclusion.

$\vdash \forall R \forall S \forall A \forall B (R \subseteq S \ \& \ A \subseteq B \Rightarrow (R \upharpoonright A) \subseteq (S \upharpoonright B))$		i
R, S, A, B	, ! 1 (Prem)	i
$R \subseteq S \ \& \ A \subseteq B$, ! 2 (Prem)	i
x, y	, ! 3 (Prem)	i
$(R \upharpoonright A)[x, y]$, ! 4 (Prem)	i
$((R \upharpoonright A)[x, y] \Rightarrow R[x, y] \ \& \ A[x])$, ! 5 ($\forall E$: P3)	i
$(R \upharpoonright A)[x, y] \Rightarrow R[x, y] \ \& \ A[x]$, ! 6 ($(\Rightarrow)E$: 5)	i
$R[x, y] \ \& \ A[x]$, ! 7 ($\Rightarrow E$: 4, 6)	i
$R[x, y]$, ! 8 ($\&E$: 7)	i
$R \subseteq S$, ! 9 ($\&E$: 2)	i
$R[x, y] \ \& \ R \subseteq S$, ! 10 ($\&I$: 8, 9)	i

$(R[x,y] \ \& \ R \subseteq S \Rightarrow S[x,y])$,! 11 ($\forall E$: C1.2)	i
$R[x,y] \ \& \ R \subseteq S \Rightarrow S[x,y]$,! 12 ($()E$: 11)	i
$S[x,y]$,! 13 ($\Rightarrow E$: 10,12)	i
$A[x]$,! 14 ($\&E$: 7)	i
$A \subseteq B$,! 15 ($\&E$: 2)	i
$A[x] \ \& \ A \subseteq B$,! 16 ($\&I$: 14,15)	i
$(A[x] \ \& \ A \subseteq B \Rightarrow B[x])$,! 17 ($\forall E$ III.2)	i
$A[x] \ \& \ A \subseteq B \Rightarrow B[x]$,! 18 ($()E$: 17)	i
$B[x]$,! 19 ($\Rightarrow E$: 16,18)	i
$S[x,y] \ \& \ B[x]$,! 20 ($\&I$: 13,19)	i
$(S[x,y] \ \& \ B[x] \Rightarrow (S \upharpoonright B)[x,y])$,! 21 ($\forall E$: P4)	i
$S[x,y] \ \& \ B[x] \Rightarrow (S \upharpoonright B)[x,y]$,! 22 ($()E$: 21)	i
$(S \upharpoonright B)[x,y]$,! 23 ($\Rightarrow E$: 20,22)	i
$(R \upharpoonright A)[x,y] \Rightarrow (S \upharpoonright B)[x,y]$,! 24 ($\Rightarrow I$: 4,23)	i
$((R \upharpoonright A)[x,y] \Rightarrow (S \upharpoonright B)[x,y])$,! 25 ($()I$: 24)	i
$\forall x \forall y ((R \upharpoonright A)[x,y] \Rightarrow (S \upharpoonright B)[x,y])$,! 26 ($\forall I$: 3,25)	i
$(R \upharpoonright A) \subseteq (S \upharpoonright B)$,! 27 ($\I: C1.1,26)	i
$R \subseteq S \ \& \ A \subseteq B \Rightarrow (R \upharpoonright A) \subseteq (S \upharpoonright B)$,! 28 ($\Rightarrow I$: 2,27)	i
$(R \subseteq S \ \& \ A \subseteq B \Rightarrow (R \upharpoonright A) \subseteq (S \upharpoonright B))$,! 29 ($()I$: 28)	i
$\forall R \forall S \forall A \forall B (R \subseteq S \ \& \ A \subseteq B \Rightarrow (R \upharpoonright A) \subseteq (S \upharpoonright B))$! 30 ($\forall I$: 1,29)	i

□

! 9.

$\vdash \forall R \forall S \forall A (R \subseteq S \Rightarrow (R \upharpoonright A) \subseteq (S \upharpoonright A))$		i
R, S, A	,! 1 (Prem)	i
$R \subseteq S$,! 2 (Prem)	i
$A \subseteq A$,! 3 ($\forall E$: III.4)	i

$R \subseteq S \ \& \ A \subseteq A$,! 4 (&I: 2,3)	i
$(R \subseteq S \ \& \ A \subseteq A \Rightarrow (R \upharpoonright A) \subseteq (S \upharpoonright A))$,! 5 (\forall E: P8)	i
$R \subseteq S \ \& \ A \subseteq A \Rightarrow (R \upharpoonright A) \subseteq (S \upharpoonright A)$,! 6 (()E: 5)	i
$(R \upharpoonright A) \subseteq (S \upharpoonright A)$,! 7 (\Rightarrow E: 4,6)	i
$R \subseteq S \Rightarrow (R \upharpoonright A) \subseteq (S \upharpoonright A)$,! 8 (\Rightarrow I: 2,7)	i
$(R \subseteq S \Rightarrow (R \upharpoonright A) \subseteq (S \upharpoonright A))$,! 9 (()I: 8)	i
$\forall R \forall S \forall A (R \subseteq S \Rightarrow (R \upharpoonright A) \subseteq (S \upharpoonright A))$! 10 (\forall I: 1,9)	i
\square		

! 10.

$\vdash \forall R \forall A \forall B (A \subseteq B \Rightarrow (R \upharpoonright A) \subseteq (R \upharpoonright B))$		i
R, A, B	,! 1 (Prem)	i
$A \subseteq B$,! 2 (Prem)	i
$R \subseteq R$,! 3 (\forall E: C1.3)	i
$R \subseteq R \ \& \ A \subseteq B$,! 4 (&I: 2,3)	i
$(R \subseteq R \ \& \ A \subseteq B \Rightarrow (R \upharpoonright A) \subseteq (R \upharpoonright B))$,! 5 (\forall E: P8)	i
$R \subseteq R \ \& \ A \subseteq B \Rightarrow (R \upharpoonright A) \subseteq (R \upharpoonright B)$,! 6 (()E: 5)	i
$(R \upharpoonright A) \subseteq (R \upharpoonright B)$,! 7 (\Rightarrow E: 4,6)	i
$A \subseteq B \Rightarrow (R \upharpoonright A) \subseteq (R \upharpoonright B)$,! 8 (\Rightarrow I: 2,7)	i
$(A \subseteq B \Rightarrow (R \upharpoonright A) \subseteq (R \upharpoonright B))$,! 9 (()I: 8)	i
$\forall R \forall A \forall B (A \subseteq B \Rightarrow (R \upharpoonright A) \subseteq (R \upharpoonright B))$! 10 (\forall I: 1,9)	i
\square		

! 11. Restrictions maintain equivalence.

$\vdash \forall R \forall S \forall A \forall B (R \equiv S \ \& \ A \equiv B \Rightarrow (R \upharpoonright A) \equiv (S \upharpoonright B))$		i
R, S, A, B	,! 1 (Prem)	i
$R \equiv S \ \& \ A \equiv B$,! 2 (Prem)	i
$R \equiv S$,! 3 (&E: 2)	i

$(R \equiv S \Rightarrow R \subseteq S \ \& \ S \subseteq R)$,! 4 ($\forall E$: C1.11) ;
 $R \equiv S \Rightarrow R \subseteq S \ \& \ S \subseteq R$,! 5 ($()E$: 4) ;
 $R \subseteq S \ \& \ S \subseteq R$,! 6 ($\Rightarrow E$: 3,5) ;
 $R \subseteq S$,! 7 ($\&E$: 6) ;
 $S \subseteq R$,! 8 ($\&E$: 6) ;
 $A \equiv B$,! 9 ($\&E$: 2) ;
 $(A \equiv B \Rightarrow A \subseteq B \ \& \ B \subseteq A)$,! 10 ($\forall E$: III.13) ;
 $A \equiv B \Rightarrow A \subseteq B \ \& \ B \subseteq A$,! 11 ($()E$: 10) ;
 $A \subseteq B \ \& \ B \subseteq A$,! 12 ($\Rightarrow E$: 9,11) ;
 $A \subseteq B$,! 13 ($\&E$: 12) ;
 $B \subseteq A$,! 14 ($\&E$: 12) ;
 $R \subseteq S \ \& \ A \subseteq B$,! 15 ($\&I$: 7,13) ;
 $(R \subseteq S \ \& \ A \subseteq B \Rightarrow (R \uparrow A) \subseteq (S \uparrow B))$,! 16 ($\forall E$: P8) ;
 $R \subseteq S \ \& \ A \subseteq B \Rightarrow (R \uparrow A) \subseteq (S \uparrow B)$,! 17 ($()E$: 16) ;
 $(R \uparrow A) \subseteq (S \uparrow B)$,! 18 ($\Rightarrow E$: 15,17) ;
 $S \subseteq R \ \& \ B \subseteq A$,! 19 ($\&I$: 8,14) ;
 $(S \subseteq R \ \& \ B \subseteq A \Rightarrow (S \uparrow B) \subseteq (R \uparrow A))$,! 20 ($\forall E$: P8) ;
 $S \subseteq R \ \& \ B \subseteq A \Rightarrow (S \uparrow B) \subseteq (R \uparrow A)$,! 21 ($()E$: 20) ;
 $(S \uparrow B) \subseteq (R \uparrow A)$,! 22 ($\Rightarrow E$: 19,21) ;
 $(R \uparrow A) \subseteq (S \uparrow B) \ \& \ (S \uparrow B) \subseteq (R \uparrow A)$,! 23 ($\&I$: 18,22) ;
 $((R \uparrow A) \subseteq (S \uparrow B) \ \& \ (S \uparrow B) \subseteq (R \uparrow A) \Rightarrow (R \uparrow A) \equiv (S \uparrow B))$,! 24 ($\forall E$: C1.6) ;
 $(R \uparrow A) \subseteq (S \uparrow B) \ \& \ (S \uparrow B) \subseteq (R \uparrow A) \Rightarrow (R \uparrow A) \equiv (S \uparrow B)$,! 25 ($()E$: 24) ;
 $(R \uparrow A) \equiv (S \uparrow B)$,! 26 ($\Rightarrow E$: 23,25) ;
 $R \equiv S \ \& \ A \equiv B \Rightarrow (R \uparrow A) \equiv (S \uparrow B)$,! 27 ($\Rightarrow I$: 2,26) ;

$(R \equiv S \ \& \ A \equiv B \Rightarrow (R \supset A) \equiv (S \supset B))$,! 28 (()I: 27) i
 $\forall R \forall S \forall A \forall B (R \equiv S \ \& \ A \equiv B \Rightarrow (R \supset A) \equiv (S \supset B))$
 ! 29 (\forall I: 1,28) i

□

! 12.

$\vdash \forall R \forall S \forall A (R \equiv S \Rightarrow (R \supset A) \equiv (S \supset A))$ i
 R, S, A ,! 1 (Prem) i
 $R \equiv S$,! 2 (Prem) i
 $A \equiv A$,! 3 (\forall E: II1.9) i
 $R \equiv S \ \& \ A \equiv A$,! 4 ($\&$ I: 2,3) i
 $(R \equiv S \ \& \ A \equiv A \Rightarrow (R \supset A) \equiv (S \supset A))$
 ,! 5 (\forall E: P11) i
 $R \equiv S \ \& \ A \equiv A \Rightarrow (R \supset A) \equiv (S \supset A)$,! 6 (()E: 5) i
 $(R \supset A) \equiv (S \supset A)$,! 7 (\Rightarrow E: 4,6) i
 $R \equiv S \Rightarrow (R \supset A) \equiv (S \supset A)$,! 8 (\Rightarrow I: 2,7) i
 $(R \equiv S \Rightarrow (R \supset A) \equiv (S \supset A))$,! 9 (()I: 8) i
 $\forall R \forall S \forall A (R \equiv S \Rightarrow (R \supset A) \equiv (S \supset A))$! 10 (\forall I: 1,9) i

□

! 13.

$\vdash \forall R \forall A \forall B (A \equiv B \Rightarrow (R \supset A) \equiv (R \supset B))$ i
 R, A, B ,! 1 (Prem) i
 $A \equiv B$,! 2 (Prem) i
 $R \equiv R$,! 3 (\forall E: C1.7) i
 $R \equiv R \ \& \ A \equiv B$,! 4 ($\&$ I: 2,3) i
 $(R \equiv R \ \& \ A \equiv B \Rightarrow (R \supset A) \equiv (R \supset B))$
 ,! 5 (\forall E: P11) i
 $R \equiv R \ \& \ A \equiv B \Rightarrow (R \supset A) \equiv (R \supset B)$,! 6 (()E: 5) i
 $(R \supset A) \equiv (R \supset B)$,! 7 (\Rightarrow E: 4,6) i

$A \equiv B \Rightarrow (R \upharpoonright A) \equiv (R \upharpoonright B)$,! 8 (\Rightarrow I: 2,7) i
 $(A \equiv B \Rightarrow (R \upharpoonright A) \equiv (R \upharpoonright B))$,! 9 ((\Rightarrow) I: 8) i
 $\forall R \forall A \forall B (A \equiv B \Rightarrow (R \upharpoonright A) \equiv (R \upharpoonright B))$! 10 (\forall I: 1,9) i
 \square

! 14. Successive restrictions is equivalent to a single restriction of an intersection. i

$\vdash \forall R \forall A \forall B ((R \upharpoonright A) \upharpoonright B) \equiv (R \upharpoonright (A \cap B))$ i
R, A, B ,! 1 (Prem) i
x, y ,! 2 (Prem) i
 $((R \upharpoonright A)[x, y] \Leftrightarrow R[x, y] \ \& \ A[x])$,! 3 (\forall E: P2) i
 $(R \upharpoonright A)[x, y] \Leftrightarrow R[x, y] \ \& \ A[x]$,! 4 ((\Leftrightarrow) E: 3) i
 $((R \upharpoonright A) \upharpoonright B)[x, y] \Leftrightarrow (R \upharpoonright A)[x, y] \ \& \ B[x]$,! 5 (\forall E: P2) i
 $((R \upharpoonright A) \upharpoonright B)[x, y] \Leftrightarrow (R \upharpoonright A)[x, y] \ \& \ B[x]$,! 6 ((\Leftrightarrow) E: 5) i
 $((R \upharpoonright (A \cap B))[x, y] \Leftrightarrow R[x, y] \ \& \ (A \cap B)[x])$,! 7 (\forall E: P2) i
 $(R \upharpoonright (A \cap B))[x, y] \Leftrightarrow R[x, y] \ \& \ (A \cap B)[x]$,! 8 ((\Leftrightarrow) E: 7) i
 $((A \cap B)[x] \Leftrightarrow A[x] \ \& \ B[x])$,! 9 (\forall E: II3.2) i
 $(A \cap B)[x] \Leftrightarrow A[x] \ \& \ B[x]$,! 10 ((\Leftrightarrow) E: 9) i
 $((R \upharpoonright A) \upharpoonright B)[x, y]$,! 11 (Prem) i
 $((R \upharpoonright A) \upharpoonright B)[x, y] \Rightarrow (R \upharpoonright A)[x, y] \ \& \ B[x]$,! 12 (\Leftrightarrow E: 6) i
 $(R \upharpoonright A)[x, y] \ \& \ B[x]$,! 13 (\Rightarrow E: 11,12) i
 $(R \upharpoonright A)[x, y]$,! 14 ($\&$ E: 13) i
 $(R \upharpoonright A)[x, y] \Rightarrow R[x, y] \ \& \ A[x]$,! 15 (\Leftrightarrow E: 4) i
 $R[x, y] \ \& \ A[x]$,! 16 (\Rightarrow E: 14,15) i
 $A[x]$,! 17 ($\&$ E: 16) i
 $B[x]$,! 18 ($\&$ E: 13) i
 $A[x] \ \& \ B[x]$,! 19 ($\&$ I: 17,18) i

$A[x] \ \& \ B[x] \Rightarrow (A \cap B)[x]$,! 20 ($\Leftrightarrow E$: 10) ;
 $(A \cap B)[x]$,! 21 ($\Rightarrow E$: 19,20) ;
 $R[x,y]$,! 22 ($\& E$: 16) ;
 $R[x,y] \ \& \ (A \cap B)[x]$,! 23 ($\& I$: 21,22) ;
 $R[x,y] \ \& \ (A \cap B)[x] \Rightarrow (R \sqsupset (A \cap B))[x,y]$,! 24 ($\Leftrightarrow E$: 8) ;
 $(R \sqsupset (A \cap B))[x,y]$,! 25 ($\Rightarrow E$: 23,24) ;
 $((R \sqsupset A) \sqsupset B)[x,y] \Rightarrow (R \sqsupset (A \cap B))[x,y]$,! 26 ($\Rightarrow I$: 11,25) ;
 $(R \sqsupset (A \cap B))[x,y]$,! 27 (Prem) ;
 $(R \sqsupset (A \cap B))[x,y] \Rightarrow R[x,y] \ \& \ (A \cap B)[x]$,! 28 ($\Leftrightarrow E$: 8) ;
 $R[x,y] \ \& \ (A \cap B)[x]$,! 29 ($\Rightarrow E$: 27,28) ;
 $(A \cap B)[x]$,! 30 ($\& E$: 29) ;
 $(A \cap B)[x] \Rightarrow A[x] \ \& \ B[x]$,! 31 ($\Leftrightarrow E$: 10) ;
 $A[x] \ \& \ B[x]$,! 32 ($\Rightarrow E$: 30,31) ;
 $A[x]$,! 33 ($\& E$: 32) ;
 $R[x,y]$,! 34 ($\& E$: 29) ;
 $R[x,y] \ \& \ A[x]$,! 35 ($\& I$: 33,34) ;
 $R[x,y] \ \& \ A[x] \Rightarrow (R \sqsupset A)[x,y]$,! 36 ($\Leftrightarrow E$: 4) ;
 $(R \sqsupset A)[x,y]$,! 37 ($\Rightarrow E$: 35,36) ;
 $B[x]$,! 38 ($\& E$: 32) ;
 $(R \sqsupset A)[x,y] \ \& \ B[x]$,! 39 ($\& I$: 37,38) ;
 $(R \sqsupset A)[x,y] \ \& \ B[x] \Rightarrow ((R \sqsupset A) \sqsupset B)[x,y]$,! 40 ($\Leftrightarrow E$: 6) ;
 $((R \sqsupset A) \sqsupset B)[x,y]$,! 41 ($\Rightarrow E$: 39,40) ;
 $(R \sqsupset (A \cap B))[x,y] \Rightarrow ((R \sqsupset A) \sqsupset B)[x,y]$,! 42 ($\Rightarrow I$: 27,41) ;
 $((R \sqsupset A) \sqsupset B)[x,y] \Leftrightarrow (R \sqsupset (A \cap B))[x,y]$,! 43 ($\Leftrightarrow I$: 26,42) ;

$((R \uparrow A) \uparrow B)_{[x,y]} \Leftrightarrow (R \uparrow (A \cap B))_{[x,y]}$,! 44 ((I: 43) i
 $\forall x \forall y ((R \uparrow A) \uparrow B)_{[x,y]} \Leftrightarrow (R \uparrow (A \cap B))_{[x,y]}$,! 45 (\forall I: 2,44) i
 $((R \uparrow A) \uparrow B) \equiv (R \uparrow (A \cap B))$,! 46 (\equiv I: C1.5,45) i
 $\forall R \forall A \forall B ((R \uparrow A) \uparrow B) \equiv (R \uparrow (A \cap B))$! 47 (\forall I: 1,45) i
 \square
! 15. i
 $\vdash \forall R \forall A \forall B (B \subseteq A \Rightarrow ((R \uparrow A) \uparrow B) \equiv (R \uparrow B))$ i
R, A, B ,! 1 (Prem) i
B \subseteq A ,! 2 (Prem) i
 $((R \uparrow A) \uparrow B) \equiv (R \uparrow (A \cap B))$,! 3 (\forall E: P14) i
 $(B \subseteq A \Rightarrow (A \cap B) \equiv B)$,! 4 (\forall E: II3.26) i
B \subseteq A \Rightarrow (A \cap B) \equiv B ,! 5 ((E: 4) i
(A \cap B) \equiv B ,! 6 (\Rightarrow E: 2,5) i
 $((A \cap B) \equiv B \Rightarrow (R \uparrow (A \cap B)) \equiv (R \uparrow B))$,! 7 (\forall E: P13) i
(A \cap B) \equiv B \Rightarrow (R \uparrow (A \cap B)) \equiv (R \uparrow B) ,! 8 ((E: 7) i
(R \uparrow (A \cap B)) \equiv (R \uparrow B) ,! 9 (\Rightarrow E: 6,8) i
 $((R \uparrow A) \uparrow B) \equiv (R \uparrow (A \cap B)) \ \& \ (R \uparrow (A \cap B)) \equiv (R \uparrow B)$,! 10 (&I: 3,9) i
 $((R \uparrow A) \uparrow B) \equiv (R \uparrow (A \cap B)) \ \& \ (R \uparrow (A \cap B)) \equiv (R \uparrow B)$
 $\Rightarrow ((R \uparrow A) \uparrow B) \equiv (R \uparrow B)$,! 11 (\forall E: C1.15) i
 $((R \uparrow A) \uparrow B) \equiv (R \uparrow (A \cap B)) \ \& \ (R \uparrow (A \cap B)) \equiv (R \uparrow B)$
 $\Rightarrow ((R \uparrow A) \uparrow B) \equiv (R \uparrow B)$,! 12 ((E: 11) i
(R \uparrow A) \uparrow B \equiv (R \uparrow B) ,! 13 (\Rightarrow E: 10,12) i
B \subseteq A \Rightarrow ((R \uparrow A) \uparrow B) \equiv (R \uparrow B) ,! 14 (\Rightarrow I: 2,13) i
 $(B \subseteq A \Rightarrow ((R \uparrow A) \uparrow B) \equiv (R \uparrow B))$,! 15 ((I: 14) i

$\forall R \forall A \forall B (B \subseteq A \Rightarrow ((R \upharpoonright A) \upharpoonright B) \equiv (R \upharpoonright B))$! 16 ($\forall I$: 1,15) i

□

! 16. i

$\vdash \forall R \forall A \forall x (A[x] \Rightarrow ((R \upharpoonright A) \upharpoonright (x^\bullet)) \equiv (R \upharpoonright (x^\bullet)))$ i

R, A, x ,! 1 (Prem) i

A[x] ,! 2 (Prem) i

$(A[x] \Rightarrow (x^\bullet) \subseteq A)$,! 3 ($\forall E$: II8.13) i

$A[x] \Rightarrow (x^\bullet) \subseteq A$,! 4 ($(\Rightarrow)E$: 3) i

$(x^\bullet) \subseteq A$,! 5 ($\Rightarrow E$: 2,4) i

$((x^\bullet) \subseteq A \Rightarrow ((R \upharpoonright A) \upharpoonright (x^\bullet)) \equiv (R \upharpoonright (x^\bullet)))$,! 6 ($\forall E$: P15) i

$(x^\bullet) \subseteq A \Rightarrow ((R \upharpoonright A) \upharpoonright (x^\bullet)) \equiv (R \upharpoonright (x^\bullet))$,! 7 ($(\Rightarrow)E$: 6) i

$((R \upharpoonright A) \upharpoonright (x^\bullet)) \equiv (R \upharpoonright (x^\bullet))$,! 8 ($\Rightarrow E$: 5,7) i

$A[x] \Rightarrow ((R \upharpoonright A) \upharpoonright (x^\bullet)) \equiv (R \upharpoonright (x^\bullet))$,! 9 ($\Rightarrow I$: 2,8) i

$(A[x] \Rightarrow ((R \upharpoonright A) \upharpoonright (x^\bullet)) \equiv (R \upharpoonright (x^\bullet)))$,! 10 ($(\Rightarrow)I$: 9) i

$\forall R \forall A \forall x (A[x] \Rightarrow ((R \upharpoonright A) \upharpoonright (x^\bullet)) \equiv (R \upharpoonright (x^\bullet)))$! 11 ($\forall I$: 1,10) i

□

! P17 and P18 establish that the restriction operator distributes over unions, around inclusion. i

! 17. i

$\vdash \forall R \forall S \forall A ((R \sqcup S) \upharpoonright A) \subseteq ((R \upharpoonright A) \sqcup (S \upharpoonright A))$ i

R, S, A ,! 1 (Prem) i

x, y ,! 2 (Prem) i

$((R \sqcup S) \upharpoonright A)[x, y]$,! 3 (Prem) i

$(((R \sqcup S) \upharpoonright A)[x, y] \Rightarrow (R \sqcup S)[x, y] \& A[x])$,! 4 ($\forall E$: P3) i

$((R \sqcup S) \upharpoonright A)[x, y] \Rightarrow (R \sqcup S)[x, y] \& A[x]$,! 5 ($(\Rightarrow)E$: 4) i

$(R \sqcup S)[x,y] \ \& \ A[x]$,! 6 (\Rightarrow E: 3,5)	i
$(R \sqcup S)[x,y]$,! 7 ($\&$ E: 6)	i
$A[x]$,! 8 ($\&$ E: 6)	i
$((R \sqcup S)[x,y] \Rightarrow R[x,y] \vee S[x,y])$,! 9 (\forall E: C2.3)	i
$(R \sqcup S)[x,y] \Rightarrow R[x,y] \vee S[x,y]$,! 10 ($(\)$ E: 9)	i
$R[x,y] \vee S[x,y]$,! 11 (\Rightarrow E: 7,10)	i
$R[x,y]$,! 12 (Prem)	i
$R[x,y] \ \& \ A[x]$,! 13 ($\&$ I: 8,12)	i
$(R[x,y] \ \& \ A[x] \Rightarrow (R \sqsupset A)[x,y])$,! 14 (\forall E: P4)	i
$R[x,y] \ \& \ A[x] \Rightarrow (R \sqsupset A)[x,y]$,! 15 ($(\)$ E: 14)	i
$(R \sqsupset A)[x,y]$,! 16 (\Rightarrow E: 13,15)	i
$(R \sqsupset A)[x,y] \vee (S \sqsupset A)[x,y]$,! 17 (\vee I: 16)	i
$R[x,y] \Rightarrow (R \sqsupset A)[x,y] \vee (S \sqsupset A)[x,y]$,! 18 (\Rightarrow I: 12,17)	i
$S[x,y]$,! 19 (Prem)	i
$S[x,y] \ \& \ A[x]$,! 20 ($\&$ I: 8,19)	i
$(S[x,y] \ \& \ A[x] \Rightarrow (S \sqsupset A)[x,y])$,! 21 (\forall E: P4)	i
$S[x,y] \ \& \ A[x] \Rightarrow (S \sqsupset A)[x,y]$,! 22 ($(\)$ E: 21)	i
$(S \sqsupset A)[x,y]$,! 23 (\Rightarrow E: 20,22)	i
$(R \sqsupset A)[x,y] \vee (S \sqsupset A)[x,y]$,! 24 (\vee I: 23)	i
$S[x,y] \Rightarrow (R \sqsupset A)[x,y] \vee (S \sqsupset A)[x,y]$,! 25 (\Rightarrow I: 19,24)	i
$(R \sqsupset A)[x,y] \vee (S \sqsupset A)[x,y]$,! 26 (\vee E: 11,18,25)	i
$((R \sqsupset A)[x,y] \vee (S \sqsupset A)[x,y] \Rightarrow ((R \sqsupset A) \sqcup (S \sqsupset A))[x,y])$,! 27 (\forall E: C2.4)	i
$(R \sqsupset A)[x,y] \vee (S \sqsupset A)[x,y] \Rightarrow ((R \sqsupset A) \sqcup (S \sqsupset A))[x,y]$,! 28 ($(\)$ E: 27)	i
$((R \sqsupset A) \sqcup (S \sqsupset A))[x,y]$,! 29 (\Rightarrow E: 26,28)	i

$((R \sqcup S) \sqsupseteq A)[x,y] \Rightarrow ((R \sqsupseteq A) \sqcup (S \sqsupseteq A))[x,y]$
!, 30 (\Rightarrow I: 3,29) i

$((R \sqcup S) \sqsupseteq A)[x,y] \Rightarrow ((R \sqsupseteq A) \sqcup (S \sqsupseteq A))[x,y]$
!, 31 ($()$ I: 30) i

$\forall x \forall y ((R \sqcup S) \sqsupseteq A)[x,y] \Rightarrow ((R \sqsupseteq A) \sqcup (S \sqsupseteq A))[x,y]$
!, 32 (\forall I: 2,31) i

$((R \sqcup S) \sqsupseteq A) \sqsubseteq ((R \sqsupseteq A) \sqcup (S \sqsupseteq A))$,! 33 (\S I: C1.1,32) i

$\forall R \forall S \forall A ((R \sqcup S) \sqsupseteq A) \sqsubseteq ((R \sqsupseteq A) \sqcup (S \sqsupseteq A))$
! 34 (\forall I: 1,33) i

□

! 18. i

$\vdash \forall R \forall S \forall A ((R \sqsupseteq A) \sqcup (S \sqsupseteq A)) \sqsubseteq ((R \sqcup S) \sqsupseteq A)$ i

R, S, A ,! 1 (Prem) i

$R \sqsubseteq (R \sqcup S)$,! 2 (\forall E: C2.7) i

$(R \sqsubseteq (R \sqcup S) \Rightarrow (R \sqsupseteq A) \sqsubseteq ((R \sqcup S) \sqsupseteq A))$
!, 3 (\forall E: P9) i

$R \sqsubseteq (R \sqcup S) \Rightarrow (R \sqsupseteq A) \sqsubseteq ((R \sqcup S) \sqsupseteq A)$,! 4 ($()$ E: 3) i

$(R \sqsupseteq A) \sqsubseteq ((R \sqcup S) \sqsupseteq A)$,! 5 (\Rightarrow E: 2,4) i

$S \sqsubseteq (R \sqcup S)$,! 6 (\forall E: C2.8) i

$(S \sqsubseteq (R \sqcup S) \Rightarrow (S \sqsupseteq A) \sqsubseteq ((R \sqcup S) \sqsupseteq A))$
!, 7 (\forall E: P9) i

$S \sqsubseteq (R \sqcup S) \Rightarrow (S \sqsupseteq A) \sqsubseteq ((R \sqcup S) \sqsupseteq A)$,! 8 ($()$ E: 7) i

$(S \sqsupseteq A) \sqsubseteq ((R \sqcup S) \sqsupseteq A)$,! 9 (\Rightarrow E: 6,8) i

$(R \sqsupseteq A) \sqsubseteq ((R \sqcup S) \sqsupseteq A) \ \& \ (S \sqsupseteq A) \sqsubseteq ((R \sqcup S) \sqsupseteq A)$
!, 10 ($\&$ I: 5,9) i

$(R \sqsupseteq A) \sqsubseteq ((R \sqcup S) \sqsupseteq A) \ \& \ (S \sqsupseteq A) \sqsubseteq ((R \sqcup S) \sqsupseteq A)$
 $\Rightarrow ((R \sqsupseteq A) \sqcup (S \sqsupseteq A)) \sqsubseteq ((R \sqcup S) \sqsupseteq A)$
!, 11 (\forall E: C2.9) i

$(R \sqsupseteq A) \sqsubseteq ((R \sqcup S) \sqsupseteq A) \ \& \ (S \sqsupseteq A) \sqsubseteq ((R \sqcup S) \sqsupseteq A)$
 $\Rightarrow ((R \sqsupseteq A) \sqcup (S \sqsupseteq A)) \sqsubseteq ((R \sqcup S) \sqsupseteq A)$
!, 12 ($()$ E: 11) i

$((R \sqsupseteq A) \sqcup (S \sqsupseteq A)) \sqsubseteq ((R \sqcup S) \sqsupseteq A)$,! 13 (\Rightarrow E: 10,12) i

$\forall R \forall S \forall A ((R \sqcup A) \sqcup (S \sqcup A)) \subseteq ((R \sqcup S) \sqcup A)$! 14 ($\forall I$: 1,13) i

□

! 19. The restriction operator distributes over unions, around equivalence. i

$\vdash \forall R \forall S \forall A ((R \sqcup S) \sqcup A) \equiv ((R \sqcup A) \sqcup (S \sqcup A))$ i

R, S, A ,! 1 (Prem) i

$((R \sqcup S) \sqcup A) \subseteq ((R \sqcup A) \sqcup (S \sqcup A))$,! 2 ($\forall E$: P17) i

$((R \sqcup A) \sqcup (S \sqcup A)) \subseteq ((R \sqcup S) \sqcup A)$,! 3 ($\forall E$: P18) i

$((R \sqcup S) \sqcup A) \subseteq ((R \sqcup A) \sqcup (S \sqcup A))$
 $\& ((R \sqcup A) \sqcup (S \sqcup A)) \subseteq ((R \sqcup S) \sqcup A)$
 ,! 4 ($\&I$: 2,3) i

$((R \sqcup S) \sqcup A) \subseteq ((R \sqcup A) \sqcup (S \sqcup A))$
 $\& ((R \sqcup A) \sqcup (S \sqcup A)) \subseteq ((R \sqcup S) \sqcup A)$
 $\Rightarrow ((R \sqcup S) \sqcup A) \equiv ((R \sqcup A) \sqcup (S \sqcup A))$
 ,! 5 ($\forall E$: C1.6) i

$((R \sqcup S) \sqcup A) \subseteq ((R \sqcup A) \sqcup (S \sqcup A))$
 $\& ((R \sqcup A) \sqcup (S \sqcup A)) \subseteq ((R \sqcup S) \sqcup A)$
 $\Rightarrow ((R \sqcup S) \sqcup A) \equiv ((R \sqcup A) \sqcup (S \sqcup A))$
 ,! 6 ($(\equiv)E$: 5) i

$((R \sqcup S) \sqcup A) \equiv ((R \sqcup A) \sqcup (S \sqcup A))$,! 7 ($\Rightarrow E$: 4,6) i

$\forall R \forall S \forall A ((R \sqcup S) \sqcup A) \equiv ((R \sqcup A) \sqcup (S \sqcup A))$! 8 ($\forall I$: 1,7) i

□

! 20. i

$\vdash \forall A (\Phi \sqcup A) \equiv \Phi$ i

A ,! 1 (Prem) i

$(\Phi \sqcup A) \subseteq \Phi$,! 2 ($\forall E$: P7) i

$((\Phi \sqcup A) \subseteq \Phi \Rightarrow (\Phi \sqcup A) \equiv \Phi)$,! 3 ($\forall E$: C4.7) i

$(\Phi \sqcup A) \subseteq \Phi \Rightarrow (\Phi \sqcup A) \equiv \Phi$,! 4 ($(\equiv)E$: 3) i

$(\Phi \sqcup A) \equiv \Phi$,! 5 ($\Rightarrow E$: 2,4) i

$\forall A (\Phi \sqcup A) \equiv \Phi$! 6 ($\forall I$: 1,5) i

□

! 21.

$\vdash \forall R (R \vdash \phi) \equiv \Phi$

R ,! 1 (Prem) i

$(\forall x \forall y \neg (R \vdash \phi)[x, y] \Rightarrow (R \vdash \phi) \equiv \Phi)$,! 2 ($\forall E$: C4.8) i

$\forall x \forall y \neg (R \vdash \phi)[x, y] \Rightarrow (R \vdash \phi) \equiv \Phi$,! 3 ($(\neg)E$: 2) i

x, y ,! 4 (Prem) i

$(R \vdash \phi)[x, y]$,! 5 (Prem) i

$(\neg (R \vdash \phi)[x, y] \Rightarrow \phi[x])$,! 6 ($\forall E$: P6) i

$(R \vdash \phi)[x, y] \Rightarrow \phi[x]$,! 7 ($(\neg)E$: 6) i

$\phi[x]$,! 8 ($\Rightarrow E$: 5,7) i

$\neg \phi[x]$,! 9 ($\forall E$: II5.3) i

\mathfrak{F} ,! 10 ($\mathfrak{F}I$: 8,9) i

$(R \vdash \phi)[x, y] \Rightarrow \mathfrak{F}$,! 11 ($\Rightarrow I$: 5,10) i

$\neg (R \vdash \phi)[x, y]$,! 12 ($\neg I$: 11) i

$\forall x \forall y \neg (R \vdash \phi)[x, y]$,! 13 ($\forall I$: 4,12) i

$(R \vdash \phi) \equiv \Phi$,! 14 ($\Rightarrow E$: 3) i

$\forall R (R \vdash \phi) \equiv \Phi$! 15 ($\forall I$: 1,14) i

□

! 22.

$\vdash \forall R \forall A (A \equiv \phi \Rightarrow (R \vdash A) \equiv \Phi)$

R, A ,! 1 (Prem) i

$A \equiv \phi$,! 2 (Prem) i

$(A \equiv \phi \Rightarrow (R \vdash A) \equiv (R \vdash \phi))$,! 3 ($\forall E$: P12) i

$A \equiv \phi \Rightarrow (R \vdash A) \equiv (R \vdash \phi)$,! 4 ($(\neg)E$: 3) i

$(R \vdash A) \equiv (R \vdash \phi)$,! 5 ($\Rightarrow E$: 2,4) i

$(R \vdash \phi) \equiv \Phi$,! 6 ($\forall E$: P21) i

$(R \vdash A) \equiv (R \vdash \phi) \ \& \ (R \vdash \phi) \equiv \Phi$,! 7 ($\&I$: 5,6) i

$((R \vdash A) \equiv (R \vdash \phi) \ \& \ (R \vdash \phi) \equiv \Phi \Rightarrow (R \vdash A) \equiv \Phi)$,! 8 ($\forall E$: C1.15)	i
$(R \vdash A) \equiv (R \vdash \phi) \ \& \ (R \vdash \phi) \equiv \Phi \Rightarrow (R \vdash A) \equiv \Phi$,! 9 ($(\)E$: 8)	i
$(R \vdash A) \equiv \Phi$,! 10 ($\Rightarrow E$: 7,9)	i
$A \equiv \phi \Rightarrow (R \vdash A) \equiv \Phi$,! 11 ($\Rightarrow I$: 2,10)	i
$(A \equiv \phi \Rightarrow (R \vdash A) \equiv \Phi)$,! 12 ($(\)I$: 11)	i
$\forall R \forall A (A \equiv \phi \Rightarrow (R \vdash A) \equiv \Phi)$! 13 ($\forall I$: 1,12)	i
\square		
! 23.		i
$\vdash \forall R \forall A ((R^D) \subseteq A \Rightarrow R \subseteq (R \vdash A))$		i
R, A	,! 1 (Prem)	i
$(R^D) \subseteq A$,! 2 (Prem)	i
x, y	,! 3 (Prem)	i
$R[x, y]$,! 4 (Prem)	i
$R[x, y] \ \& \ (R^D) \subseteq A$,! 5 ($\&I$: 2,4)	i
$(R[x, y] \ \& \ (R^D) \subseteq A \Rightarrow A[x])$,! 6 ($\forall E$: C5.6)	i
$R[x, y] \ \& \ (R^D) \subseteq A \Rightarrow A[x]$,! 7 ($(\)E$: 6)	i
$A[x]$,! 8 ($\Rightarrow E$: 5,7)	i
$R[x, y] \ \& \ A[x]$,! 9 ($\&I$: 4,8)	i
$(R[x, y] \ \& \ A[x] \Rightarrow (R \vdash A)[x, y])$,! 10 ($\forall E$: P4)	i
$R[x, y] \ \& \ A[x] \Rightarrow (R \vdash A)[x, y]$,! 11 ($(\)E$: 10)	i
$(R \vdash A)[x, y]$,! 12 ($\Rightarrow E$: 9,11)	i
$R[x, y] \Rightarrow (R \vdash A)[x, y]$,! 13 ($\Rightarrow I$: 4,12)	i
$(R[x, y] \Rightarrow (R \vdash A)[x, y])$,! 14 ($(\)I$: 13)	i
$\forall x \forall y (R[x, y] \Rightarrow (R \vdash A)[x, y])$,! 15 ($\forall I$: 3)	i
$R \subseteq (R \vdash A)$,! 16 ($\$I$: C1.1,15)	i
$(R^D) \subseteq A \Rightarrow R \subseteq (R \vdash A)$,! 17 ($\Rightarrow I$: 2,16)	i

$((R^D) \subseteq A \Rightarrow R \subseteq (R \uparrow A))$,! 18 ((I: 17) i
 $\forall R \forall A ((R^D) \subseteq A \Rightarrow R \subseteq (R \uparrow A))$! 19 (\forall I: 1,18) i
 \square
! 24. i
 $\vdash \forall R \forall A ((R^D) \subseteq A \Rightarrow (R \uparrow A) \equiv R)$ i
R, A ,! 1 (Prem) i
 $(R \uparrow A) \subseteq R$,! 2 (\forall E: P7) i
 $(R^D) \subseteq A$,! 3 (Prem) i
 $((R^D) \subseteq A \Rightarrow R \subseteq (R \uparrow A))$,! 4 (\forall E: P23) i
 $(R^D) \subseteq A \Rightarrow R \subseteq (R \uparrow A)$,! 5 ((E: 4) i
 $R \subseteq (R \uparrow A)$,! 6 (\Rightarrow E: 3,5) i
 $(R \uparrow A) \subseteq R \ \& \ R \subseteq (R \uparrow A)$,! 7 ($\&$ I: 2,6) i
 $((R \uparrow A) \subseteq R \ \& \ R \subseteq (R \uparrow A) \Rightarrow (R \uparrow A) \equiv R)$,! 8 (\forall E: C1.6) i
 $(R \uparrow A) \subseteq R \ \& \ R \subseteq (R \uparrow A) \Rightarrow (R \uparrow A) \equiv R$,! 9 ((E: 8) i
 $(R \uparrow A) \equiv R$,! 10 (\Rightarrow E: 7,9) i
 $(R^D) \subseteq A \Rightarrow (R \uparrow A) \equiv R$,! 11 (\Rightarrow I: 3,10) i
 $((R^D) \subseteq A \Rightarrow (R \uparrow A) \equiv R)$,! 12 ((I: 11) i
 $\forall R \forall A ((R^D) \subseteq A \Rightarrow (R \uparrow A) \equiv R)$! 13 (\forall I: 1,12) i

\square

! 25. i

$\vdash \forall R \forall S \forall A (R \subseteq S \ \& \ (R^D) \subseteq A \Rightarrow R \subseteq (S \uparrow A))$ i
R, S, A ,! 1 (Prem) i
 $R \subseteq S \ \& \ (R^D) \subseteq A$,! 2 (Prem) i
 $R \subseteq S$,! 3 ($\&$ E: 2) i
 $(R \subseteq S \Rightarrow (R \uparrow A) \subseteq (S \uparrow A))$,! 4 (\forall E: P9) i
 $R \subseteq S \Rightarrow (R \uparrow A) \subseteq (S \uparrow A)$,! 5 ((E: 4) i

$(R \upharpoonright A) \subseteq (S \upharpoonright A)$,! 6 ($\Rightarrow E$: 3,5)	i
$(R^D) \subseteq A$,! 7 ($\&E$: 2)	i
$((R^D) \subseteq A \Rightarrow R \subseteq (R \upharpoonright A))$,! 8 ($\forall E$: P23)	i
$(R^D) \subseteq A \Rightarrow R \subseteq (R \upharpoonright A)$,! 9 ($(\Rightarrow)E$: 8)	i
$R \subseteq (R \upharpoonright A)$,! 10 ($\Rightarrow E$: 7,9)	i
$R \subseteq (R \upharpoonright A) \ \& \ (R \upharpoonright A) \subseteq (S \upharpoonright A)$,! 11 ($\&I$: 6,10)	i
$(R \subseteq (R \upharpoonright A) \ \& \ (R \upharpoonright A) \subseteq (S \upharpoonright A) \Rightarrow R \subseteq (S \upharpoonright A))$,! 12 ($\forall E$: C1.4)	i
$R \subseteq (R \upharpoonright A) \ \& \ (R \upharpoonright A) \subseteq (S \upharpoonright A) \Rightarrow R \subseteq (S \upharpoonright A)$,! 13 ($(\Rightarrow)E$: 12)	i
$R \subseteq (S \upharpoonright A)$,! 14 ($\Rightarrow E$: 11,13)	i
$R \subseteq S \ \& \ (R^D) \subseteq A \Rightarrow R \subseteq (S \upharpoonright A)$,! 15 ($\Rightarrow I$: 2,14)	i
$(R \subseteq S \ \& \ (R^D) \subseteq A \Rightarrow R \subseteq (S \upharpoonright A))$,! 16 ($(\Rightarrow)I$: 15)	i
$\forall R \forall S \forall A (R \subseteq S \ \& \ (R^D) \subseteq A \Rightarrow R \subseteq (S \upharpoonright A))$! 17 ($\forall I$: 1,16)	i

□

! 26.

$\vdash \forall R \forall A \forall B ((R^D) \subseteq (A \cup B) \Rightarrow ((R \upharpoonright A) \sqcup (R \upharpoonright B)) \equiv R)$		i
R, A, B	,! 1 (Prem)	i
$(R^D) \subseteq (A \cup B)$,! 2 (Prem)	i
$(R \upharpoonright A) \subseteq R$,! 3 ($\forall E$: P7)	i
$(R \upharpoonright B) \subseteq R$,! 4 ($\forall E$: P7)	i
$(R \upharpoonright A) \subseteq R \ \& \ (R \upharpoonright B) \subseteq R$,! 5 ($\&I$: 3,4)	i
$((R \upharpoonright A) \subseteq R \ \& \ (R \upharpoonright B) \subseteq R \Rightarrow ((R \upharpoonright A) \sqcup (R \upharpoonright B)) \subseteq R)$,! 6 ($\forall E$: C2.9)	i
$(R \upharpoonright A) \subseteq R \ \& \ (R \upharpoonright B) \subseteq R \Rightarrow ((R \upharpoonright A) \sqcup (R \upharpoonright B)) \subseteq R$,! 7 ($(\Rightarrow)E$: 6)	i
$((R \upharpoonright A) \sqcup (R \upharpoonright B)) \subseteq R$,! 8 ($\Rightarrow E$: 5,7)	i
x, y	,! 9 (Prem)	i
$R[x, y]$,! 10 (Prem)	i

$R[x,y] \ \& \ (R^D) \subseteq (A \cup B)$,! 11 (&I: 2,10) ;
 $(R[x,y] \ \& \ (R^D) \subseteq (A \cup B) \Rightarrow (A \cup B)[x])$,! 12 ($\forall E$: C5.6) ;
 $R[x,y] \ \& \ (R^D) \subseteq (A \cup B) \Rightarrow (A \cup B)[x]$,! 13 ($(\)E$: 12) ;
 $(A \cup B)[x]$,! 14 ($\Rightarrow E$: 11,13) ;
 $((A \cup B)[x] \Rightarrow A[x] \vee B[x])$,! 15 ($\forall E$: II2.3) ;
 $(A \cup B)[x] \Rightarrow A[x] \vee B[x]$,! 16 ($(\)E$: 15) ;
 $A[x] \vee B[x]$,! 17 ($\Rightarrow E$: 14,16) ;
 $A[x]$,! 18 (Prem) ;
 $R[x,y] \ \& \ A[x]$,! 19 (&I: 10,18) ;
 $(R[x,y] \ \& \ A[x] \Rightarrow (R \lceil A)[x,y])$,! 20 ($\forall E$: P4) ;
 $R[x,y] \ \& \ A[x] \Rightarrow (R \lceil A)[x,y]$,! 21 ($(\)E$: 20) ;
 $(R \lceil A)[x,y]$,! 22 ($\Rightarrow E$: 19,21) ;
 $(R \lceil A)[x,y] \vee (R \lceil B)[x,y]$,! 23 ($\vee I$: 22) ;
 $A[x] \Rightarrow (R \lceil A)[x,y] \vee (R \lceil B)[x,y]$,! 24 ($\Rightarrow I$: 18,23) ;
 $B[x]$,! 25 (Prem) ;
 $R[x,y] \ \& \ B[x]$,! 26 (&I: 10,25) ;
 $(R[x,y] \ \& \ B[x] \Rightarrow (R \lceil B)[x,y])$,! 27 ($\forall E$: P4) ;
 $R[x,y] \ \& \ B[x] \Rightarrow (R \lceil B)[x,y]$,! 28 ($(\)E$: 27) ;
 $(R \lceil B)[x,y]$,! 29 ($\Rightarrow E$: 26,28) ;
 $(R \lceil A)[x,y] \vee (R \lceil B)[x,y]$,! 30 ($\vee I$: 29) ;
 $B[x] \Rightarrow (R \lceil A)[x,y] \vee (R \lceil B)[x,y]$,! 31 ($\Rightarrow I$: 25,30) ;
 $(R \lceil A)[x,y] \vee (R \lceil B)[x,y]$,! 32 ($\vee E$: 17,24,31) ;

$$\begin{aligned} & ((R \uparrow A)[x,y] \vee (R \uparrow B)[x,y] \\ & \Rightarrow ((R \uparrow A) \sqcup (R \uparrow B))[x,y]) \\ & \qquad \qquad \qquad ,! 33 (\forall E: C2.4) \quad i \end{aligned}$$

$$\begin{aligned} & (R \uparrow A)[x,y] \vee (R \uparrow B)[x,y] \Rightarrow ((R \uparrow A) \sqcup (R \uparrow B))[x,y] \\ & \qquad \qquad \qquad ,! 34 (())E: 33) \quad i \end{aligned}$$

$$\begin{aligned} & ((R \uparrow A) \sqcup (R \uparrow B))[x,y] \\ & \qquad \qquad \qquad ,! 35 (\Rightarrow E: 32,34) \quad i \end{aligned}$$

$$R[x,y] \Rightarrow ((R \uparrow A) \sqcup (R \uparrow B))[x,y] \quad ,! 36 (\Rightarrow I: 10,35) \quad i$$

$$\begin{aligned} & (R[x,y] \Rightarrow ((R \uparrow A) \sqcup (R \uparrow B))[x,y]) \\ & \qquad \qquad \qquad ,! 37 (())I: 36) \quad i \end{aligned}$$

$$\begin{aligned} & \forall x \forall y (R[x,y] \Rightarrow ((R \uparrow A) \sqcup (R \uparrow B))[x,y]) \\ & \qquad \qquad \qquad ,! 38 (\forall I: 9,37) \quad i \end{aligned}$$

$$R^c \subseteq ((R \uparrow A) \sqcup (R \uparrow B)) \quad ,! 39 (\$I: C1.1,38) \quad i$$

$$\begin{aligned} & ((R \uparrow A) \sqcup (R \uparrow B)) \subseteq R \ \& \ R^c \subseteq ((R \uparrow A) \sqcup (R \uparrow B)) \\ & \qquad \qquad \qquad ,! 40 (\&I: 8,39) \quad i \end{aligned}$$

$$\begin{aligned} & (((R \uparrow A) \sqcup (R \uparrow B)) \subseteq R \ \& \ R^c \subseteq ((R \uparrow A) \sqcup (R \uparrow B)) \\ & \Rightarrow ((R \uparrow A) \sqcup (R \uparrow B)) \equiv R) \\ & \qquad \qquad \qquad ,! 41 (\forall E: C1.6) \quad i \end{aligned}$$

$$\begin{aligned} & ((R \uparrow A) \sqcup (R \uparrow B)) \subseteq R \ \& \ R^c \subseteq ((R \uparrow A) \sqcup (R \uparrow B)) \\ & \Rightarrow ((R \uparrow A) \sqcup (R \uparrow B)) \equiv R \\ & \qquad \qquad \qquad ,! 42 (())E: 41) \quad i \end{aligned}$$

$$(R \uparrow A) \sqcup (R \uparrow B) \equiv R \quad ,! 43 (\Rightarrow E: 40,42) \quad i$$

$$\begin{aligned} & (R^D) \subseteq (A \cup B) \Rightarrow ((R \uparrow A) \sqcup (R \uparrow B)) \equiv R \\ & \qquad \qquad \qquad ,! 44 (\Rightarrow I: 2,43) \quad i \end{aligned}$$

$$\begin{aligned} & ((R^D) \subseteq (A \cup B) \Rightarrow ((R \uparrow A) \sqcup (R \uparrow B)) \equiv R) \\ & \qquad \qquad \qquad ,! 45 (())I: 44) \quad i \end{aligned}$$

$$\begin{aligned} & \forall R \forall A \forall B ((R^D) \subseteq (A \cup B) \Rightarrow ((R \uparrow A) \sqcup (R \uparrow B)) \equiv R) \\ & \qquad \qquad \qquad ! 46 (\forall I: 1,45) \quad i \end{aligned}$$

□

! 27. i

$$\vdash \forall R \forall A \forall B ((R^D) \equiv (A \cup B) \Rightarrow ((R \uparrow A) \sqcup (R \uparrow B)) \equiv R) \quad i$$

$$R, A, B \quad ,! 1 (\text{Prem}) \quad i$$

$$(R^D) \equiv (A \cup B) \quad ,! 2 (\text{Prem}) \quad i$$

$$((R^D) \equiv (A \cup B) \Rightarrow (R^D) \subseteq (A \cup B)) ,! 3 (\forall E: II1.11) \quad i$$

$(R^D) \equiv (A \cup B) \Rightarrow (R^D) \subseteq (A \cup B)$,! 4 ((E: 3) i

$(R^D) \subseteq (A \cup B)$,! 5 (\Rightarrow E: 2,4) i

$((R^D) \subseteq (A \cup B) \Rightarrow ((R \upharpoonright A) \sqcup (R \upharpoonright B)) \equiv R)$
,! 6 (\forall E: P26) i

$(R^D) \subseteq (A \cup B) \Rightarrow ((R \upharpoonright A) \sqcup (R \upharpoonright B)) \equiv R$
,! 7 ((E: 6) i

$((R \upharpoonright A) \sqcup (R \upharpoonright B)) \equiv R$,! 8 (\Rightarrow E: 5,7) i

$(R^D) \equiv (A \cup B) \Rightarrow ((R \upharpoonright A) \sqcup (R \upharpoonright B)) \equiv R$
,! 9 (\Rightarrow I: 2,8) i

$((R^D) \equiv (A \cup B) \Rightarrow ((R \upharpoonright A) \sqcup (R \upharpoonright B)) \equiv R)$
,! 10 ((I: 9) i

$\forall R \forall A \forall B ((R^D) \equiv (A \cup B) \Rightarrow ((R \upharpoonright A) \sqcup (R \upharpoonright B)) \equiv R)$
! 11 (\forall I: 1,10) i

□

! 28. i

$\vdash \forall R \forall A ((R \upharpoonright A)^D \subseteq (R^D)$ i

R, A ,! 1 (Prem) i

$(R \upharpoonright A) \subseteq R$,! 2 (\forall E: P7) i

$((R \upharpoonright A) \subseteq R \Rightarrow ((R \upharpoonright A)^D \subseteq (R^D))$,! 3 (\forall E C5.14) i

$(R \upharpoonright A) \subseteq R \Rightarrow ((R \upharpoonright A)^D \subseteq (R^D)$,! 4 ((E: 3) i

$((R \upharpoonright A)^D \subseteq (R^D)$,! 5 (\Rightarrow E: 2,4) i

$\forall R \forall A ((R \upharpoonright A)^D \subseteq (R^D)$! 6 (\forall I: 1,5) i

□

! 29. i

$\vdash \forall R \forall A ((R \upharpoonright A)^D \subseteq A$ i

R, A ,! 1 (Prem) i

$(\forall x \forall y ((R \upharpoonright A)[x,y] \Rightarrow A[x]) \Rightarrow ((R \upharpoonright A)^D \subseteq A)$
,! 2 (\forall E: C5.11) i

$\forall x \forall y ((R \upharpoonright A)[x,y] \Rightarrow A[x]) \Rightarrow ((R \upharpoonright A)^D \subseteq A$
,! 3 ((E: 2) i

$\forall x \forall y ((R \uparrow A)[x,y] \Rightarrow A[x])$,! 4 ($\forall E$: P6)	i
$((R \uparrow A)^D) \subseteq A$,! 5 ($\Rightarrow E$: 3,4)	i
$\forall R \forall A ((R \uparrow A)^D) \subseteq A$! 6 ($\forall I$: 1,5)	i
\square		
! 30.		i
$\vdash \forall R \forall A ((R \uparrow A)^D) \equiv ((R^D) \cap A)$		i
R, A	,! 1 (Prem)	i
$((R \uparrow A)^D) \subseteq (R^D)$,! 2 ($\forall E$: P28)	i
$((R \uparrow A)^D) \subseteq A$,! 3 ($\forall E$: P29)	i
$((R \uparrow A)^D) \subseteq (R^D) \ \& \ ((R \uparrow A)^D) \subseteq A$,! 4 ($\&I$: 2,3)	i
$(((R \uparrow A)^D) \subseteq (R^D) \ \& \ ((R \uparrow A)^D) \subseteq A$ $\Rightarrow ((R \uparrow A)^D) \subseteq ((R^D) \cap A))$,! 5 ($\forall E$: II3.12)	i
$((R \uparrow A)^D) \subseteq (R^D) \ \& \ ((R \uparrow A)^D) \subseteq A$ $\Rightarrow ((R \uparrow A)^D) \subseteq ((R^D) \cap A)$,! 6 ($(\)E$: 5)	i
$((R \uparrow A)^D) \subseteq ((R^D) \cap A)$,! 7 ($\Rightarrow E$: 4,6)	i
$(\forall x (((R^D) \cap A)[x] \Rightarrow \exists y (R \uparrow A)[x,y])$ $\Rightarrow ((R^D) \cap A) \subseteq ((R \uparrow A)^D))$,! 8 ($\forall E$: C5.12)	i
$\forall x (((R^D) \cap A)[x] \Rightarrow \exists y (R \uparrow A)[x,y])$ $\Rightarrow ((R^D) \cap A) \subseteq ((R \uparrow A)^D)$,! 9 ($(\)E$: 8)	i
x	,! 10 (Prem)	i
$((R^D) \cap A)[x]$,! 11 (Prem)	i
$(((R^D) \cap A)[x] \Rightarrow (R^D)[x] \ \& \ A[x])$,! 12 ($\forall E$: II3.3)	i
$((R^D) \cap A)[x] \Rightarrow (R^D)[x] \ \& \ A[x]$,! 13 ($(\)E$: 12)	i
$(R^D)[x] \ \& \ A[x]$,! 14 ($\Rightarrow E$: 11,13)	i
$(R^D)[x]$,! 15 ($\&E$: 14)	i
$A[x]$,! 16 ($\&E$: 14)	i

$((R^D)[\mathbf{x}] \Rightarrow \exists y R[\mathbf{x}, y])$,! 17 ($\forall E$: C5.3)	i
$(R^D)[\mathbf{x}] \Rightarrow \exists y R[\mathbf{x}, y]$,! 18 ($(\Rightarrow)E$: 17)	i
$\exists y R[\mathbf{x}, y]$,! 19 ($\Rightarrow E$: 15,18)	i
$R[\mathbf{x}, y]$,! 20 ($\exists E$: 19)	i
$R[\mathbf{x}, y] \ \& \ A[\mathbf{x}]$,! 21 ($\&I$: 16,20)	i
$(R[\mathbf{x}, y] \ \& \ A[\mathbf{x}] \Rightarrow (R \lceil A)[\mathbf{x}, y])$,! 22 ($\forall E$: P4)	i
$R[\mathbf{x}, y] \ \& \ A[\mathbf{x}] \Rightarrow (R \lceil A)[\mathbf{x}, y]$,! 23 ($(\Rightarrow)E$: 22)	i
$(R \lceil A)[\mathbf{x}, y]$,! 24 ($\Rightarrow E$: 21,23)	i
$\exists y (R \lceil A)[\mathbf{x}, y]$,! 25 ($\exists I$: 24)	i
$((R^D) \cap A)[\mathbf{x}] \Rightarrow \exists y (R \lceil A)[\mathbf{x}, y]$,! 26 ($\Rightarrow I$: 11,25)	i
$(((R^D) \cap A)[\mathbf{x}] \Rightarrow \exists y (R \lceil A)[\mathbf{x}, y])$,! 27 ($(\Rightarrow)I$: 26)	i
$\forall x (((R^D) \cap A)[x] \Rightarrow \exists y (R \lceil A)[x, y])$,! 28 ($\forall I$: 10,27)	i
$((R^D) \cap A) \subseteq ((R \lceil A)^D)$,! 29 ($\Rightarrow E$: 9,28)	i
$((R \lceil A)^D) \subseteq ((R^D) \cap A) \ \& \ ((R^D) \cap A) \subseteq ((R \lceil A)^D)$,! 30 ($\&I$: 7,29)	i
$(((R \lceil A)^D) \subseteq ((R^D) \cap A) \ \& \ ((R^D) \cap A) \subseteq ((R \lceil A)^D) \Rightarrow ((R \lceil A)^D) \equiv ((R^D) \cap A))$,! 31 ($\forall E$: III.1.8)	i
$((R \lceil A)^D) \subseteq ((R^D) \cap A) \ \& \ ((R^D) \cap A) \subseteq ((R \lceil A)^D) \Rightarrow ((R \lceil A)^D) \equiv ((R^D) \cap A)$,! 32 ($(\Rightarrow)E$: 31)	i
$((R \lceil A)^D) \equiv ((R^D) \cap A)$,! 33 ($\Rightarrow E$: 30,32)	i
$\forall R \forall A ((R \lceil A)^D) \equiv ((R^D) \cap A)$! 34 ($\forall I$: 1,33)	i

□

! 31.

$\vdash \forall R \forall A (A \subseteq (R^D) \Rightarrow ((R \lceil A)^D) \equiv A)$		i
R, A	,! 1 (Prem)	i
$A \subseteq (R^D)$,! 2 (Prem)	i
$(A \subseteq (R^D) \Rightarrow ((R^D) \cap A) \equiv A)$,! 3 ($\forall E$: II3.26)	i

$\mathbf{A} \subseteq (\mathbf{R}^D) \Rightarrow ((\mathbf{R}^D) \cap \mathbf{A}) \equiv \mathbf{A}$,! 4 ((E: 3) i
 $((\mathbf{R}^D) \cap \mathbf{A}) \equiv \mathbf{A}$,! 5 (\Rightarrow E: 2,4) i
 $((\mathbf{R} \sqsupset \mathbf{A})^D) \equiv ((\mathbf{R}^D) \cap \mathbf{A})$,! 6 (\forall E: P30) i
 $((\mathbf{R} \sqsupset \mathbf{A})^D) \equiv ((\mathbf{R}^D) \cap \mathbf{A}) \ \& \ ((\mathbf{R}^D) \cap \mathbf{A}) \equiv \mathbf{A}$
, ! 7 (&I: 5,6) i
 $(((\mathbf{R} \sqsupset \mathbf{A})^D) \equiv ((\mathbf{R}^D) \cap \mathbf{A}) \ \& \ ((\mathbf{R}^D) \cap \mathbf{A}) \equiv \mathbf{A}$
 $\Rightarrow ((\mathbf{R} \sqsupset \mathbf{A})^D) \equiv ((\mathbf{R}^D) \cap \mathbf{A})$)
, ! 8 (\forall E: III.15) i
 $((\mathbf{R} \sqsupset \mathbf{A})^D) \equiv ((\mathbf{R}^D) \cap \mathbf{A}) \ \& \ ((\mathbf{R}^D) \cap \mathbf{A}) \equiv \mathbf{A}$
 $\Rightarrow ((\mathbf{R} \sqsupset \mathbf{A})^D) \equiv ((\mathbf{R}^D) \cap \mathbf{A})$
, ! 9 ((E: 8) i
 $((\mathbf{R} \sqsupset \mathbf{A})^D) \equiv ((\mathbf{R}^D) \cap \mathbf{A})$,! 10 (\Rightarrow E: 7,9) i
 $\mathbf{A} \subseteq (\mathbf{R}^D) \Rightarrow ((\mathbf{R} \sqsupset \mathbf{A})^D) \equiv \mathbf{A}$,! 11 (\Rightarrow I: 2,10) i
 $(\mathbf{A} \subseteq (\mathbf{R}^D) \Rightarrow ((\mathbf{R} \sqsupset \mathbf{A})^D) \equiv \mathbf{A})$,! 12 ((I: 11) i
 $\forall \mathbf{R} \forall \mathbf{A} (\mathbf{A} \subseteq (\mathbf{R}^D) \Rightarrow ((\mathbf{R} \sqsupset \mathbf{A})^D) \equiv \mathbf{A})$! 13 (\forall I: 1,12) i

□

! 32. i

$\vdash \forall \mathbf{R} \forall \mathbf{A} \forall \mathbf{B} ((\mathbf{R}^D) \equiv \mathbf{A} \ \& \ \mathbf{B} \subseteq \mathbf{A} \Rightarrow ((\mathbf{R} \sqsupset \mathbf{B})^D) \equiv \mathbf{B})$ i
 $\mathbf{R}, \mathbf{A}, \mathbf{B}$,! 1 (Prem) i
 $(\mathbf{R}^D) \equiv \mathbf{A} \ \& \ \mathbf{B} \subseteq \mathbf{A}$,! 2 (Prem) i
 $(\mathbf{R}^D) \equiv \mathbf{A}$,! 3 (&E: 2) i
 $((\mathbf{R}^D) \equiv \mathbf{A} \Rightarrow \mathbf{A} \subseteq (\mathbf{R}^D))$,! 4 (\forall E: III.12) i
 $(\mathbf{R}^D) \equiv \mathbf{A} \Rightarrow \mathbf{A} \subseteq (\mathbf{R}^D)$,! 5 ((E: 4) i
 $\mathbf{A} \subseteq (\mathbf{R}^D)$,! 6 (\Rightarrow E: 3,5) i
 $\mathbf{B} \subseteq \mathbf{A}$,! 7 (&E: 2) i
 $\mathbf{B} \subseteq \mathbf{A} \ \& \ \mathbf{A} \subseteq (\mathbf{R}^D)$,! 8 (&I: 6,7) i
 $(\mathbf{B} \subseteq \mathbf{A} \ \& \ \mathbf{A} \subseteq (\mathbf{R}^D) \Rightarrow \mathbf{B} \subseteq (\mathbf{R}^D))$,! 9 (\forall E: III.5) i
 $\mathbf{B} \subseteq \mathbf{A} \ \& \ \mathbf{A} \subseteq (\mathbf{R}^D) \Rightarrow \mathbf{B} \subseteq (\mathbf{R}^D)$,! 10 ((E: 9) i
 $\mathbf{B} \subseteq (\mathbf{R}^D)$,! 11 (\Rightarrow E: 8,10) i

$(B \subseteq (R^D) \Rightarrow ((R \uparrow B)^D) \equiv B)$,! 12 ($\forall E$: P31) i
 $B \subseteq (R^D) \Rightarrow ((R \uparrow B)^D) \equiv B$,! 13 ($(\)E$: 12) i
 $((R \uparrow B)^D) \equiv B$,! 14 ($\Rightarrow E$: 11,13) i
 $(R^D) \equiv A \ \& \ B \subseteq A \Rightarrow ((R \uparrow B)^D) \equiv B$,! 15 ($\Rightarrow I$: 2,14) i
 $((R^D) \equiv A \ \& \ B \subseteq A \Rightarrow ((R \uparrow B)^D) \equiv B)$,! 16 ($(\)I$: 15) i
 $\forall R \forall A \forall B ((R^D) \equiv A \ \& \ B \subseteq A \Rightarrow ((R \uparrow B)^D) \equiv B)$! 17 ($\forall I$: 1,16) i

□

! 33. i

$\vdash \forall R \forall A (((R^D) \cap A) \equiv \phi \Rightarrow (R \uparrow A) \equiv \Phi)$ i
R, A ,! 1 (Prem) i
 $((R^D) \cap A) \equiv \phi$,! 2 (Prem) i
 $((R \uparrow A)^D) \equiv ((R^D) \cap A)$,! 3 ($\forall E$: P30) i
 $((R \uparrow A)^D) \equiv ((R^D) \cap A) \ \& \ ((R^D) \cap A) \equiv \phi$,! 4 ($\&I$: 2,3) i
 $(((R \uparrow A)^D) \equiv ((R^D) \cap A) \ \& \ ((R^D) \cap A) \equiv \phi$
 $\Rightarrow ((R \uparrow A)^D) \equiv \phi)$,! 5 ($\forall E$: II1.15) i
 $((R \uparrow A)^D) \equiv ((R^D) \cap A) \ \& \ ((R^D) \cap A) \equiv \phi \Rightarrow ((R \uparrow A)^D) \equiv \phi$,! 6 ($(\)E$: 5) i
 $((R \uparrow A)^D) \equiv \phi$,! 7 ($\Rightarrow E$: 4,6) i
 $(((R \uparrow A)^D) \equiv \phi \Rightarrow (R \uparrow A) \equiv \Phi)$,! 8 ($\forall E$: C5.24) i
 $((R \uparrow A)^D) \equiv \phi \Rightarrow (R \uparrow A) \equiv \Phi$,! 9 ($(\)E$: 8) i
 $(R \uparrow A) \equiv \Phi$,! 10 ($\Rightarrow E$: 7,9) i
 $((R^D) \cap A) \equiv \phi \Rightarrow (R \uparrow A) \equiv \Phi$,! 11 ($\Rightarrow I$: 2,10) i
 $(((R^D) \cap A) \equiv \phi \Rightarrow (R \uparrow A) \equiv \Phi)$,! 12 ($(\)I$: 11) i
 $\forall R \forall A (((R^D) \cap A) \equiv \phi \Rightarrow (R \uparrow A) \equiv \Phi)$! 13 ($\forall I$: 1,12) i

□

! 34. i

$\vdash \forall R \forall S \forall A (((R^D) \cap A) \equiv \phi \Rightarrow ((R \sqcup S) \uparrow A) \equiv (S \uparrow A))$;
R, S, A ,! 1 (Prem) ;
 $((R^D) \cap A) \equiv \phi$,! 2 (Prem) ;
 $(((R^D) \cap A) \equiv \phi \Rightarrow (R \uparrow A) \equiv \Phi)$,! 3 ($\forall E$: P33) ;
 $((R^D) \cap A) \equiv \phi \Rightarrow (R \uparrow A) \equiv \Phi$,! 4 ($(\Rightarrow E)$: 3) ;
 $(R \uparrow A) \equiv \Phi$,! 5 ($\Rightarrow E$: 2,4) ;
 $((R \uparrow A) \equiv \Phi \Rightarrow ((R \uparrow A) \sqcup (S \uparrow A)) \equiv (S \uparrow A))$,! 6 ($\forall E$: C4.13) ;
 $(R \uparrow A) \equiv \Phi \Rightarrow ((R \uparrow A) \sqcup (S \uparrow A)) \equiv (S \uparrow A)$,! 7 ($(\Rightarrow E)$: 6) ;
 $((R \uparrow A) \sqcup (S \uparrow A)) \equiv (S \uparrow A)$,! 8 ($\Rightarrow E$: 5,7) ;
 $((R \sqcup S) \uparrow A) \equiv ((R \uparrow A) \sqcup (S \uparrow A))$,! 9 ($\forall E$: P19) ;
 $((R \sqcup S) \uparrow A) \equiv ((R \uparrow A) \sqcup (S \uparrow A))$
 $\& ((R \uparrow A) \sqcup (S \uparrow A)) \equiv (S \uparrow A)$,! 10 ($\&I$: 8,9) ;
 $(((R \sqcup S) \uparrow A) \equiv ((R \uparrow A) \sqcup (S \uparrow A))$
 $\& ((R \uparrow A) \sqcup (S \uparrow A)) \equiv (S \uparrow A)$
 $\Rightarrow ((R \sqcup S) \uparrow A) \equiv (S \uparrow A))$,! 11 ($\forall E$: C1.15) ;
 $((R \sqcup S) \uparrow A) \equiv ((R \uparrow A) \sqcup (S \uparrow A))$
 $\& ((R \uparrow A) \sqcup (S \uparrow A)) \equiv (S \uparrow A)$
 $\Rightarrow ((R \sqcup S) \uparrow A) \equiv (S \uparrow A)$,! 12 ($(\Rightarrow E)$: 11) ;
 $((R \sqcup S) \uparrow A) \equiv (S \uparrow A)$,! 13 ($\Rightarrow E$: 10,12) ;
 $((R^D) \cap A) \equiv \phi \Rightarrow ((R \sqcup S) \uparrow A) \equiv (S \uparrow A)$,! 14 ($\Rightarrow I$: 2,13) ;
 $(((R^D) \cap A) \equiv \phi \Rightarrow ((R \sqcup S) \uparrow A) \equiv (S \uparrow A))$,! 15 ($(\Rightarrow I)$: 14) ;
 $\forall R \forall S \forall A (((R^D) \cap A) \equiv \phi \Rightarrow ((R \sqcup S) \uparrow A) \equiv (S \uparrow A))$! 16 ($\forall I$: 1,15) ;

□

$\vdash \forall R \forall S \forall A (((R^D) \cap A) \equiv \phi \Rightarrow ((S \sqcup R) \uparrow A) \equiv (S \uparrow A))$;
R, S, A ,! 1 (Prem) ;
 $((R^D) \cap A) \equiv \phi$,! 2 (Prem) ;
 $(((R^D) \cap A) \equiv \phi \Rightarrow ((R \sqcup S) \uparrow A) \equiv (S \uparrow A))$;
, ! 3 ($\forall E$: P34) ;
 $((R^D) \cap A) \equiv \phi \Rightarrow ((R \sqcup S) \uparrow A) \equiv (S \uparrow A)$;
, ! 4 ($(\Rightarrow)E$: 3) ;
 $((R \sqcup S) \uparrow A) \equiv (S \uparrow A)$,! 5 ($\Rightarrow E$: 2,4) ;
 $(S \sqcup R) \equiv (R \sqcup S)$,! 6 ($\forall E$: C2.6) ;
 $((S \sqcup R) \equiv (R \sqcup S) \Rightarrow ((S \sqcup R) \uparrow A) \equiv ((R \sqcup S) \uparrow A))$;
, ! 7 ($\forall E$: P12) ;
 $(S \sqcup R) \equiv (R \sqcup S) \Rightarrow ((S \sqcup R) \uparrow A) \equiv ((R \sqcup S) \uparrow A)$;
, ! 8 ($(\Rightarrow)E$: 7) ;
 $((S \sqcup R) \uparrow A) \equiv ((R \sqcup S) \uparrow A)$,! 9 ($\Rightarrow E$: 6,8) ;
 $((S \sqcup R) \uparrow A) \equiv ((R \sqcup S) \uparrow A) \ \& \ ((R \sqcup S) \uparrow A) \equiv (S \uparrow A)$;
, ! 10 ($\&I$: 5,9) ;
 $(((S \sqcup R) \uparrow A) \equiv ((R \sqcup S) \uparrow A) \ \& \ ((R \sqcup S) \uparrow A) \equiv (S \uparrow A) \Rightarrow ((S \sqcup R) \uparrow A) \equiv (S \uparrow A))$;
, ! 11 ($\forall E$: C1.15) ;
 $((S \sqcup R) \uparrow A) \equiv ((R \sqcup S) \uparrow A) \ \& \ ((R \sqcup S) \uparrow A) \equiv (S \uparrow A) \Rightarrow ((S \sqcup R) \uparrow A) \equiv (S \uparrow A)$;
, ! 12 ($(\Rightarrow)E$: 11) ;
 $((S \sqcup R) \uparrow A) \equiv (S \uparrow A)$,! 13 ($\Rightarrow E$: 10,12) ;
 $((R^D) \cap A) \equiv \phi \Rightarrow ((S \sqcup R) \uparrow A) \equiv (S \uparrow A)$;
, ! 14 ($\Rightarrow I$: 2,13) ;
 $(((R^D) \cap A) \equiv \phi \Rightarrow ((S \sqcup R) \uparrow A) \equiv (S \uparrow A))$;
, ! 15 ($(\Rightarrow)I$: 14) ;
 $\forall R \forall S \forall A (((R^D) \cap A) \equiv \phi \Rightarrow ((S \sqcup R) \uparrow A) \equiv (S \uparrow A))$;
! 16 ($\forall I$: 1,15) ;

□

! 36.

$\vdash \forall R \forall A ((R^D) \equiv \mathbf{U} \Rightarrow ((R \uparrow A)^D) \equiv A)$;
R, A ,! 1 (Prem) ;

$(R^D) \equiv U$,! 2 (Prem) i
 $A \subseteq U$,! 3 ($\forall E$: II6.7) i
 $(R^D) \equiv U \ \& \ A \subseteq U$,! 4 ($\&I$: 2,3) i
 $((R^D) \equiv U \ \& \ A \subseteq U \Rightarrow ((R \uparrow A)^D) \equiv A)$
,! 5 ($\forall E$: P32) i
 $(R^D) \equiv U \ \& \ A \subseteq U \Rightarrow ((R \uparrow A)^D) \equiv A$,! 6 ($(\)E$: 5) i
 $((R \uparrow A)^D) \equiv A$,! 7 ($\Rightarrow E$: 4,6) i
 $(R^D) \equiv U \Rightarrow ((R \uparrow A)^D) \equiv A$,! 8 ($\Rightarrow I$: 2,7) i
 $((R^D) \equiv U \Rightarrow ((R \uparrow A)^D) \equiv A)$,! 9 ($(\)I$: 8) i
 $\forall R \forall A ((R^D) \equiv U \Rightarrow ((R \uparrow A)^D) \equiv A)$! 10 ($\forall I$: 1,9) i

□

! 37. i

$\vdash \forall R \forall x (\neg (R^D)[x] \Rightarrow (R \uparrow (x^\bullet)) \equiv \Phi)$ i
 R, x ,! 1 (Prem) i
 $\neg (R^D)[x]$,! 2 (Prem) i
 $(\neg (R^D)[x] \Rightarrow ((R^D) \cap (x^\bullet)) \equiv \phi)$,! 3 ($\forall E$: II8.40) i
 $\neg (R^D)[x] \Rightarrow ((R^D) \cap (x^\bullet)) \equiv \phi$,! 4 ($(\)E$: 3) i
 $((R^D) \cap (x^\bullet)) \equiv \phi$,! 5 ($\Rightarrow E$: 2,4) i
 $(((R^D) \cap (x^\bullet)) \equiv \phi \Rightarrow (R \uparrow (x^\bullet)) \equiv \Phi)$
,! 6 ($\forall E$: P33) i
 $((R^D) \cap (x^\bullet)) \equiv \phi \Rightarrow (R \uparrow (x^\bullet)) \equiv \Phi$,! 7 ($(\)E$: 6) i
 $(R \uparrow (x^\bullet)) \equiv \Phi$,! 8 ($\Rightarrow E$: 5,7) i
 $\neg (R^D)[x] \Rightarrow (R \uparrow (x^\bullet)) \equiv \Phi$,! 9 ($\Rightarrow I$: 2,8) i
 $(\neg (R^D)[x] \Rightarrow (R \uparrow (x^\bullet)) \equiv \Phi)$,! 10 ($(\)I$: 9) i
 $\forall R \forall x (\neg (R^D)[x] \Rightarrow (R \uparrow (x^\bullet)) \equiv \Phi)$! 11 ($\forall I$: 1,10) i

□

! 38. i

$\vdash \forall A ((\Phi \vdash A)^I) \equiv \phi$		i
A	$,! 1$ (Prem)	i
$(\Phi \vdash A) \equiv \Phi$	$,! 2$ ($\forall E$: P20)	i
$((\Phi \vdash A) \equiv \Phi \Rightarrow ((\Phi \vdash A)^I) \equiv \phi)$	$,! 3$ ($\forall E$: C6.29)	i
$(\Phi \vdash A) \equiv \Phi \Rightarrow ((\Phi \vdash A)^I) \equiv \phi$	$,! 4$ ($(\)E$: 3)	i
$((\Phi \vdash A)^I) \equiv \phi$	$,! 5$ ($\Rightarrow E$: 2,4)	i
$\forall A ((\Phi \vdash A)^I) \equiv \phi$	$! 6$ ($\forall I$: 1,5)	i

□

! 39.

$\vdash \forall R ((R \vdash \phi)^I) \equiv \phi$		i
R	$,! 1$ (Prem)	i
$(R \vdash \phi) \equiv \Phi$	$,! 2$ ($\forall E$: P21)	i
$((R \vdash \phi) \equiv \Phi \Rightarrow ((R \vdash \phi)^I) \equiv \phi)$	$,! 3$ ($\forall E$: C6.29)	i
$(R \vdash \phi) \equiv \Phi \Rightarrow ((R \vdash \phi)^I) \equiv \phi$	$,! 4$ ($(\)E$: 3)	i
$((R \vdash \phi)^I) \equiv \phi$	$,! 5$ ($\Rightarrow E$: 2,4)	i
$\forall R (R \vdash \phi) \equiv \Phi$	$! 6$ ($\forall I$: 1,5)	i

□

! 40.

$\vdash \forall R \forall A (A \equiv \phi \Rightarrow ((R \vdash A)^I) \equiv \phi)$		i
R, A	$,! 1$ (Prem)	i
$A \equiv \phi$	$,! 2$ (Prem)	i
$(A \equiv \phi \Rightarrow (R \vdash A) \equiv \Phi)$	$,! 3$ ($\forall E$: P22)	i
$A \equiv \phi \Rightarrow (R \vdash A) \equiv \Phi$	$,! 4$ ($(\)E$: 3)	i
$(R \vdash A) \equiv \Phi$	$,! 5$ ($\Rightarrow E$: 2,4)	i
$((R \vdash A) \equiv \Phi \Rightarrow ((R \vdash A)^I) \equiv \phi)$	$,! 6$ ($\forall E$: C6.29)	i
$(R \vdash A) \equiv \Phi \Rightarrow ((R \vdash A)^I) \equiv \phi$	$,! 7$ ($(\)E$: 6)	i
$((R \vdash A)^I) \equiv \phi$	$,! 8$ ($\Rightarrow E$: 5,7)	i

$\mathbf{A} \equiv \phi \Rightarrow ((\mathbf{R} \lceil \mathbf{A})^I) \equiv \phi$,! 9 (\Rightarrow I: 2,8) ;
 $(\mathbf{A} \equiv \phi \Rightarrow ((\mathbf{R} \lceil \mathbf{A})^I) \equiv \phi)$,! 10 ((I): 2,9) ;
 $\forall \mathbf{R} \forall \mathbf{A} (\mathbf{A} \equiv \phi \Rightarrow ((\mathbf{R} \lceil \mathbf{A})^I) \equiv \phi)$! 11 (\forall I: 1,10) ;
 \square
! 41. ;
 $\vdash \forall \mathbf{R} \forall \mathbf{A} \forall \mathbf{B} ((\mathbf{R}^D) \subseteq (\mathbf{A} \cup \mathbf{B})$
 $\Rightarrow ((\mathbf{R} \lceil \mathbf{A})^I) \cup ((\mathbf{R} \lceil \mathbf{B})^I) \equiv (\mathbf{R}^I))$;
 $\mathbf{R}, \mathbf{A}, \mathbf{B}$,! 1 (Prem) ;
 $(\mathbf{R}^D) \subseteq (\mathbf{A} \cup \mathbf{B})$,! 2 (Prem) ;
 $((\mathbf{R}^D) \subseteq (\mathbf{A} \cup \mathbf{B}) \Rightarrow ((\mathbf{R} \lceil \mathbf{A}) \sqcup (\mathbf{R} \lceil \mathbf{B})) \equiv \mathbf{R})$
,! 3 (\forall E: P26) ;
 $(\mathbf{R}^D) \subseteq (\mathbf{A} \cup \mathbf{B}) \Rightarrow ((\mathbf{R} \lceil \mathbf{A}) \sqcup (\mathbf{R} \lceil \mathbf{B})) \equiv \mathbf{R}$
,! 4 ((E): 3) ;
 $((\mathbf{R} \lceil \mathbf{A}) \sqcup (\mathbf{R} \lceil \mathbf{B})) \equiv \mathbf{R}$,! 5 (\Rightarrow E: 2,4) ;
 $((\mathbf{R} \lceil \mathbf{A}) \sqcup (\mathbf{R} \lceil \mathbf{B})) \equiv \mathbf{R} \Rightarrow (((\mathbf{R} \lceil \mathbf{A}) \sqcup (\mathbf{R} \lceil \mathbf{B}))^I) \equiv (\mathbf{R}^I))$
,! 6 (\forall E: C6.22) ;
 $((\mathbf{R} \lceil \mathbf{A}) \sqcup (\mathbf{R} \lceil \mathbf{B})) \equiv \mathbf{R} \Rightarrow (((\mathbf{R} \lceil \mathbf{A}) \sqcup (\mathbf{R} \lceil \mathbf{B}))^I) \equiv (\mathbf{R}^I))$
,! 7 ((E): 6) ;
 $((\mathbf{R} \lceil \mathbf{A}) \sqcup (\mathbf{R} \lceil \mathbf{B}))^I \equiv (\mathbf{R}^I)$,! 8 (\Rightarrow E: 5,7) ;
 $((\mathbf{R} \lceil \mathbf{A}) \sqcup (\mathbf{R} \lceil \mathbf{B}))^I \equiv ((\mathbf{R} \lceil \mathbf{A})^I) \cup ((\mathbf{R} \lceil \mathbf{B})^I)$
,! 9 (\forall E: C6.25) ;
 $((\mathbf{R} \lceil \mathbf{A}) \sqcup (\mathbf{R} \lceil \mathbf{B}))^I \equiv ((\mathbf{R} \lceil \mathbf{A})^I) \cup ((\mathbf{R} \lceil \mathbf{B})^I)$
& $((\mathbf{R} \lceil \mathbf{A}) \sqcup (\mathbf{R} \lceil \mathbf{B}))^I \equiv (\mathbf{R}^I)$
,! 10 (&I: 8,9) ;
 $((\mathbf{R} \lceil \mathbf{A}) \sqcup (\mathbf{R} \lceil \mathbf{B}))^I \equiv ((\mathbf{R} \lceil \mathbf{A})^I) \cup ((\mathbf{R} \lceil \mathbf{B})^I)$
& $((\mathbf{R} \lceil \mathbf{A}) \sqcup (\mathbf{R} \lceil \mathbf{B}))^I \equiv (\mathbf{R}^I)$
 $\Rightarrow ((\mathbf{R} \lceil \mathbf{A})^I) \cup ((\mathbf{R} \lceil \mathbf{B})^I) \equiv (\mathbf{R}^I)$
,! 11 (\forall E: C1.18) ;
 $((\mathbf{R} \lceil \mathbf{A}) \sqcup (\mathbf{R} \lceil \mathbf{B}))^I \equiv ((\mathbf{R} \lceil \mathbf{A})^I) \cup ((\mathbf{R} \lceil \mathbf{B})^I)$
& $((\mathbf{R} \lceil \mathbf{A}) \sqcup (\mathbf{R} \lceil \mathbf{B}))^I \equiv (\mathbf{R}^I)$
 $\Rightarrow ((\mathbf{R} \lceil \mathbf{A})^I) \cup ((\mathbf{R} \lceil \mathbf{B})^I) \equiv (\mathbf{R}^I)$
,! 12 ((E): 11) ;
 $((\mathbf{R} \lceil \mathbf{A})^I) \cup ((\mathbf{R} \lceil \mathbf{B})^I) \equiv (\mathbf{R}^I)$,! 13 (\Rightarrow E: 10,12) ;

$$(R^D) \subseteq (A \cup B) \Rightarrow (((R \uparrow A)^I) \cup ((R \uparrow B)^I)) \equiv (R^I) \quad ,! 14 (\Rightarrow I: 2,13) \quad i$$

$$((R^D) \subseteq (A \cup B) \Rightarrow (((R \uparrow A)^I) \cup ((R \uparrow B)^I)) \equiv (R^I)) \quad ,! 15 ((I): 14) \quad i$$

$$\forall R \forall A \forall B ((R^D) \subseteq (A \cup B) \Rightarrow (((R \uparrow A)^I) \cup ((R \uparrow B)^I)) \equiv (R^I)) \quad ! 16 (\forall I: 1,15) \quad i$$

□

! 42. i

$$\vdash \forall R \forall A \forall B ((R^D) \equiv (A \cup B) \Rightarrow (((R \uparrow A)^I) \cup ((R \uparrow B)^I)) \equiv (R^I)) \quad i$$

$$R, A, B \quad ,! 1 (\text{Prem}) \quad i$$

$$(R^D) \equiv (A \cup B) \quad ,! 2 (\text{Prem}) \quad i$$

$$((R^D) \equiv (A \cup B) \Rightarrow (R^D) \subseteq (A \cup B)) \quad ,! 3 (\forall E: \text{II1.11}) \quad i$$

$$(R^D) \equiv (A \cup B) \Rightarrow (R^D) \subseteq (A \cup B) \quad ,! 4 ((E): 3) \quad i$$

$$(R^D) \subseteq (A \cup B) \quad ,! 5 (\Rightarrow E: 2,4) \quad i$$

$$((R^D) \subseteq (A \cup B) \Rightarrow (((R \uparrow A)^I) \cup ((R \uparrow B)^I)) \equiv (R^I)) \quad ,! 6 (\forall E: \text{P41}) \quad i$$

$$(R^D) \subseteq (A \cup B) \Rightarrow (((R \uparrow A)^I) \cup ((R \uparrow B)^I)) \equiv (R^I) \quad ,! 7 ((E): 6) \quad i$$

$$(((R \uparrow A)^I) \cup ((R \uparrow B)^I)) \equiv (R^I) \quad ,! 8 (\Rightarrow E: 5,7) \quad i$$

$$(R^D) \equiv (A \cup B) \Rightarrow (((R \uparrow A)^I) \cup ((R \uparrow B)^I)) \equiv (R^I) \quad ,! 9 (\Rightarrow I: 2,8) \quad i$$

$$((R^D) \equiv (A \cup B) \Rightarrow (((R \uparrow A)^I) \cup ((R \uparrow B)^I)) \equiv (R^I)) \quad ,! 10 ((I): 9) \quad i$$

$$\forall R \forall A \forall B ((R^D) \equiv (A \cup B) \Rightarrow (((R \uparrow A)^I) \cup ((R \uparrow B)^I)) \equiv (R^I)) \quad ! 11 (\forall I: 1,10) \quad i$$

□

! 43. i

$$\vdash \forall R \forall A \forall B \forall C ((A \cup B) \equiv C \Rightarrow (((R \uparrow A)^I) \cup ((R \uparrow B)^I)) \equiv ((R \uparrow C)^I)) \quad i$$

R, A, B, C	,! 1 (Prem)	i
$(A \cup B) \equiv C$,! 2 (Prem)	i
$((R \lceil C)^D) \subseteq C$,! 3 ($\forall E$: P29)	i
$(A \cup B) \equiv C \ \& \ ((R \lceil C)^D) \subseteq C$,! 4 ($\&I$: 2,3)	i
$((A \cup B) \equiv C \ \& \ ((R \lceil C)^D) \subseteq C \Rightarrow ((R \lceil C)^D) \subseteq (A \cup B))$,! 5 ($\forall E$: III.31)	i
$(A \cup B) \equiv C \ \& \ ((R \lceil C)^D) \subseteq C \Rightarrow ((R \lceil C)^D) \subseteq (A \cup B)$,! 6 ($()E$: 5)	i
$((R \lceil C)^D) \subseteq (A \cup B)$,! 7 ($\Rightarrow E$: 4,6)	i
$(((R \lceil C)^D) \subseteq (A \cup B) \Rightarrow (((R \lceil C) \lceil A)^I) \cup (((R \lceil C) \lceil B)^I)) \equiv ((R \lceil C)^I))$,! 8 ($\forall E$: P41)	i
$((R \lceil C)^D) \subseteq (A \cup B) \Rightarrow (((R \lceil C) \lceil A)^I) \cup (((R \lceil C) \lceil B)^I)) \equiv ((R \lceil C)^I)$,! 9 ($()E$: 8)	i
$(((R \lceil C) \lceil A)^I) \cup (((R \lceil C) \lceil B)^I)) \equiv ((R \lceil C)^I)$,! 10 ($\Rightarrow E$: 7,9)	i
$A \subseteq (A \cup B)$,! 11 ($\forall E$: II2.12)	i
$(A \cup B) \equiv C \ \& \ A \subseteq (A \cup B)$,! 12 ($\&I$: 2,11)	i
$((A \cup B) \equiv C \ \& \ A \subseteq (A \cup B) \Rightarrow A \subseteq C)$,! 13 ($\forall E$: III.32)	i
$(A \cup B) \equiv C \ \& \ A \subseteq (A \cup B) \Rightarrow A \subseteq C$,! 14 ($()E$: 13)	i
$A \subseteq C$,! 15 ($\Rightarrow E$: 12,14)	i
$(A \subseteq C \Rightarrow ((R \lceil C) \lceil A) \equiv (R \lceil A))$,! 16 ($\forall E$: P15)	i
$A \subseteq C \Rightarrow ((R \lceil C) \lceil A) \equiv (R \lceil A)$,! 17 ($()E$: 16)	i
$((R \lceil C) \lceil A) \equiv (R \lceil A)$,! 18 ($\Rightarrow E$: 15,17)	i
$(((R \lceil C) \lceil A) \equiv (R \lceil A) \Rightarrow (((R \lceil C) \lceil A)^I) \equiv ((R \lceil A)^I))$,! 19 ($\forall E$: C6.22)	i
$((R \lceil C) \lceil A) \equiv (R \lceil A) \Rightarrow (((R \lceil C) \lceil A)^I) \equiv ((R \lceil A)^I)$,! 20 ($()E$: 19)	i
$(((R \lceil C) \lceil A)^I) \equiv ((R \lceil A)^I)$,! 21 ($\Rightarrow E$: 18,20)	i

$((\mathbf{R} \mid \mathbf{C}) \mid \mathbf{A})^{\mathbf{I}} \equiv (\mathbf{R} \mid \mathbf{A})^{\mathbf{I}}$
 $\& (((\mathbf{R} \mid \mathbf{C}) \mid \mathbf{A})^{\mathbf{I}} \cup ((\mathbf{R} \mid \mathbf{C}) \mid \mathbf{B})^{\mathbf{I}}) \equiv ((\mathbf{R} \mid \mathbf{C})^{\mathbf{I}})$
 ,! 22 (&I: 10,21) ;

$\mathbf{B} \subseteq (\mathbf{A} \cup \mathbf{B})$,! 23 (\forall E: II2.13) ;

$(\mathbf{A} \cup \mathbf{B}) \equiv \mathbf{C} \& \mathbf{B} \subseteq (\mathbf{A} \cup \mathbf{B})$,! 24 (&I: 2,23) ;

$((\mathbf{A} \cup \mathbf{B}) \equiv \mathbf{C} \& \mathbf{B} \subseteq (\mathbf{A} \cup \mathbf{B}) \Rightarrow \mathbf{B} \subseteq \mathbf{C})$
 ,! 25 (\forall E: III1.32) ;

$(\mathbf{A} \cup \mathbf{B}) \equiv \mathbf{C} \& \mathbf{B} \subseteq (\mathbf{A} \cup \mathbf{B}) \Rightarrow \mathbf{B} \subseteq \mathbf{C}$,! 26 (()E: 25) ;

$\mathbf{B} \subseteq \mathbf{C}$,! 27 (\Rightarrow E: 24,26) ;

$(\mathbf{B} \subseteq \mathbf{C} \Rightarrow ((\mathbf{R} \mid \mathbf{C}) \mid \mathbf{B}) \equiv (\mathbf{R} \mid \mathbf{B}))$,! 28 (\forall E: P15) ;

$\mathbf{B} \subseteq \mathbf{C} \Rightarrow ((\mathbf{R} \mid \mathbf{C}) \mid \mathbf{B}) \equiv (\mathbf{R} \mid \mathbf{B})$,! 29 (()E: 28) ;

$((\mathbf{R} \mid \mathbf{C}) \mid \mathbf{B}) \equiv (\mathbf{R} \mid \mathbf{B})$,! 30 (\Rightarrow E: 27,29) ;

$(((\mathbf{R} \mid \mathbf{C}) \mid \mathbf{B}) \equiv (\mathbf{R} \mid \mathbf{B}) \Rightarrow ((\mathbf{R} \mid \mathbf{C}) \mid \mathbf{B})^{\mathbf{I}} \equiv (\mathbf{R} \mid \mathbf{B})^{\mathbf{I}})$
 ,! 31 (\forall E: C6.22) ;

$((\mathbf{R} \mid \mathbf{C}) \mid \mathbf{B}) \equiv (\mathbf{R} \mid \mathbf{B}) \Rightarrow ((\mathbf{R} \mid \mathbf{C}) \mid \mathbf{B})^{\mathbf{I}} \equiv (\mathbf{R} \mid \mathbf{B})^{\mathbf{I}}$
 ,! 32 (()E: 31) ;

$((\mathbf{R} \mid \mathbf{C}) \mid \mathbf{B})^{\mathbf{I}} \equiv (\mathbf{R} \mid \mathbf{B})^{\mathbf{I}}$,! 33 (\Rightarrow E: 30,32) ;

$((\mathbf{R} \mid \mathbf{C}) \mid \mathbf{A})^{\mathbf{I}} \equiv (\mathbf{R} \mid \mathbf{A})^{\mathbf{I}}$
 $\& ((\mathbf{R} \mid \mathbf{C}) \mid \mathbf{B})^{\mathbf{I}} \equiv (\mathbf{R} \mid \mathbf{B})^{\mathbf{I}}$
 $\& (((\mathbf{R} \mid \mathbf{C}) \mid \mathbf{A})^{\mathbf{I}} \cup ((\mathbf{R} \mid \mathbf{C}) \mid \mathbf{B})^{\mathbf{I}}) \equiv ((\mathbf{R} \mid \mathbf{C})^{\mathbf{I}})$
 ,! 34 (&I: 22,33) ;

$(((\mathbf{R} \mid \mathbf{C}) \mid \mathbf{A})^{\mathbf{I}} \equiv (\mathbf{R} \mid \mathbf{A})^{\mathbf{I}}$
 $\& ((\mathbf{R} \mid \mathbf{C}) \mid \mathbf{B})^{\mathbf{I}} \equiv (\mathbf{R} \mid \mathbf{B})^{\mathbf{I}}$
 $\& (((\mathbf{R} \mid \mathbf{C}) \mid \mathbf{A})^{\mathbf{I}} \cup ((\mathbf{R} \mid \mathbf{C}) \mid \mathbf{B})^{\mathbf{I}}) \equiv ((\mathbf{R} \mid \mathbf{C})^{\mathbf{I}})$
 $\Rightarrow (((\mathbf{R} \mid \mathbf{A})^{\mathbf{I}} \cup ((\mathbf{R} \mid \mathbf{B})^{\mathbf{I}}) \equiv ((\mathbf{R} \mid \mathbf{C})^{\mathbf{I}}))$
 ,! 35 (\forall E: II2.41) ;

$((\mathbf{R} \mid \mathbf{C}) \mid \mathbf{A})^{\mathbf{I}} \equiv (\mathbf{R} \mid \mathbf{A})^{\mathbf{I}}$
 $\& ((\mathbf{R} \mid \mathbf{C}) \mid \mathbf{B})^{\mathbf{I}} \equiv (\mathbf{R} \mid \mathbf{B})^{\mathbf{I}}$
 $\& (((\mathbf{R} \mid \mathbf{C}) \mid \mathbf{A})^{\mathbf{I}} \cup ((\mathbf{R} \mid \mathbf{C}) \mid \mathbf{B})^{\mathbf{I}}) \equiv ((\mathbf{R} \mid \mathbf{C})^{\mathbf{I}})$
 $\Rightarrow (((\mathbf{R} \mid \mathbf{A})^{\mathbf{I}} \cup ((\mathbf{R} \mid \mathbf{B})^{\mathbf{I}}) \equiv ((\mathbf{R} \mid \mathbf{C})^{\mathbf{I}})$
 ,! 36 (()E: 35) ;

$(((\mathbf{R} \mid \mathbf{A})^{\mathbf{I}} \cup ((\mathbf{R} \mid \mathbf{B})^{\mathbf{I}}) \equiv ((\mathbf{R} \mid \mathbf{C})^{\mathbf{I}})$
 ,! 37 (\Rightarrow E: 35,36) ;

$$(A \cup B) \equiv C \Rightarrow ((R \lceil A)^I \cup (R \lceil B)^I) \equiv (R \lceil C)^I)$$

, ! 38 (\Rightarrow I: 2,37) i

$$((A \cup B) \equiv C \Rightarrow ((R \lceil A)^I \cup (R \lceil B)^I) \equiv (R \lceil C)^I))$$

, ! 39 ($()$ I: 38) i

$$\forall R \forall A \forall B \forall C ((A \cup B) \equiv C$$

$$\Rightarrow ((R \lceil A)^I \cup (R \lceil B)^I) \equiv (R \lceil C)^I)$$

! 40 (\forall I: 1,39) i

□

! 44. i

$$\vdash \forall R \forall S \forall x (\neg (R^D)[x] \Rightarrow ((R \sqcup S) \lceil (x^\bullet))^I \equiv (S \lceil (x^\bullet))^I) ;$$

$$R, S, x \quad , ! 1 (\text{Prem}) \quad i$$

$$\neg (R^D)[x] \quad , ! 2 (\text{Prem}) \quad i$$

$$(\neg (R^D)[x] \Rightarrow ((R^D) \cap (x^\bullet)) \equiv \phi) \quad , ! 3 (\forall E: \text{II8.40}) \quad i$$

$$\neg (R^D)[x] \Rightarrow ((R^D) \cap (x^\bullet)) \equiv \phi \quad , ! 4 ($()$ E: 3) \quad i$$

$$((R^D) \cap (x^\bullet)) \equiv \phi \quad , ! 5 (\Rightarrow E: 2,4) \quad i$$

$$(((R^D) \cap (x^\bullet)) \equiv \phi \Rightarrow ((R \sqcup S) \lceil (x^\bullet)) \equiv (S \lceil (x^\bullet)))$$

, ! 6 (\forall E: P34) i

$$((R^D) \cap (x^\bullet)) \equiv \phi \Rightarrow ((R \sqcup S) \lceil (x^\bullet)) \equiv (S \lceil (x^\bullet))$$

, ! 7 ($()$ E: 6) i

$$((R \sqcup S) \lceil (x^\bullet)) \equiv (S \lceil (x^\bullet)) \quad , ! 8 (\Rightarrow E: 5,7) \quad i$$

$$(((R \sqcup S) \lceil (x^\bullet)) \equiv (S \lceil (x^\bullet))$$

$$\Rightarrow (((R \sqcup S) \lceil (x^\bullet))^I \equiv ((S \lceil (x^\bullet))^I))$$

, ! 9 (\forall E: C6.22) i

$$((R \sqcup S) \lceil (x^\bullet)) \equiv (S \lceil (x^\bullet))$$

$$\Rightarrow (((R \sqcup S) \lceil (x^\bullet))^I \equiv ((S \lceil (x^\bullet))^I))$$

, ! 10 ($()$ E: 9) i

$$(((R \sqcup S) \lceil (x^\bullet))^I \equiv ((S \lceil (x^\bullet))^I) \quad , ! 11 (\Rightarrow E: 8,10) \quad i$$

$$\neg (R^D)[x] \Rightarrow (((R \sqcup S) \lceil (x^\bullet))^I \equiv ((S \lceil (x^\bullet))^I)$$

, ! 12 (\Rightarrow I: 2,11) i

$$(\neg (R^D)[x] \Rightarrow (((R \sqcup S) \lceil (x^\bullet))^I \equiv ((S \lceil (x^\bullet))^I))$$

, ! 13 ($()$ I: 12) i

$$\forall R \forall S \forall x (\neg (R^D)[x] \Rightarrow (((R \sqcup S) \lceil (x^\bullet))^I \equiv ((S \lceil (x^\bullet))^I))$$

! 14 (\forall I: 1,13) i

□

! 45. P45's proof copies P44's.

$\vdash \forall R \forall S \forall x (\neg (R^D)[x] \Rightarrow ((S \sqcup R) \uparrow (x^\bullet))^I \equiv ((S \uparrow (x^\bullet))^I))$;

R, S, x ,! 1 (Prem) ;

$\neg (R^D)[x]$,! 2 (Prem) ;

$(\neg (R^D)[x] \Rightarrow ((R^D) \cap (x^\bullet)) \equiv \phi)$,! 3 ($\forall E$: II8.40) ;

$\neg (R^D)[x] \Rightarrow ((R^D) \cap (x^\bullet)) \equiv \phi$,! 4 ($()E$: 3) ;

$((R^D) \cap (x^\bullet)) \equiv \phi$,! 5 ($\Rightarrow E$: 2,4) ;

$(((R^D) \cap (x^\bullet)) \equiv \phi \Rightarrow ((S \sqcup R) \uparrow (x^\bullet)) \equiv (S \uparrow (x^\bullet)))$
 ,! 6 ($\forall E$: P35) ;

$((R^D) \cap (x^\bullet)) \equiv \phi \Rightarrow ((S \sqcup R) \uparrow (x^\bullet)) \equiv (S \uparrow (x^\bullet))$
 ,! 7 ($()E$: 6) ;

$((S \sqcup R) \uparrow (x^\bullet)) \equiv (S \uparrow (x^\bullet))$,! 8 ($\Rightarrow E$: 5,7) ;

$(((S \sqcup R) \uparrow (x^\bullet)) \equiv (S \uparrow (x^\bullet))$
 $\Rightarrow (((S \sqcup R) \uparrow (x^\bullet))^I \equiv ((S \uparrow (x^\bullet))^I))$
 ,! 9 ($\forall E$: C6.22) ;

$((S \sqcup R) \uparrow (x^\bullet)) \equiv (S \uparrow (x^\bullet))$
 $\Rightarrow (((S \sqcup R) \uparrow (x^\bullet))^I \equiv ((S \uparrow (x^\bullet))^I)$
 ,! 10 ($()E$: 9) ;

$((S \sqcup R) \uparrow (x^\bullet))^I \equiv ((S \uparrow (x^\bullet))^I)$,! 11 ($\Rightarrow E$: 8,10) ;

$\neg (R^D)[x] \Rightarrow (((S \sqcup R) \uparrow (x^\bullet))^I \equiv ((S \uparrow (x^\bullet))^I)$
 ,! 12 ($\Rightarrow I$: 2,11) ;

$(\neg (R^D)[x] \Rightarrow (((S \sqcup R) \uparrow (x^\bullet))^I \equiv ((S \uparrow (x^\bullet))^I))$
 ,! 13 ($()I$: 12) ;

$\forall R \forall S \forall x (\neg (R^D)[x] \Rightarrow (((S \sqcup R) \uparrow (x^\bullet))^I \equiv ((S \uparrow (x^\bullet))^I))$
 ! 14 ($\forall I$: 1,13) ;

□

! 46. P46 is a lemma for P47.

$\vdash \forall R \forall A \forall y (((R \uparrow A)^I)[y] \Leftrightarrow \exists z (R[z,y] \& A[z]))$;

R, A, y ,! 1 (Prem) ;

$(((R \uparrow A)^I)[y] \Leftrightarrow \exists x (R \uparrow A)[x,y])$,! 2 ($\forall E$: C6.2) ;

$((R \upharpoonright A)^I)[y] \Leftrightarrow \exists x (R \upharpoonright A)[x, y]$,! 3 ((E: 2)	i
$((R \upharpoonright A)[x, y] \Leftrightarrow R[x, y] \ \& \ A[x])$,! 4 ($\forall E$: P2)	i
$(R \upharpoonright A)[x, y] \Leftrightarrow R[x, y] \ \& \ A[x]$,! 5 ((E: 4)	i
$((R \upharpoonright A)^I)[y]$,! 6 (Prem)	i
$((R \upharpoonright A)^I)[y] \Rightarrow \exists x (R \upharpoonright A)[x, y]$,! 7 ($\Leftrightarrow E$: 3)	i
$\exists x (R \upharpoonright A)[x, y]$,! 8 ($\Rightarrow E$: 6,7)	i
$(R \upharpoonright A)[x, y]$,! 9 ($\exists E$: 8)	i
$(R \upharpoonright A)[x, y] \Rightarrow R[x, y] \ \& \ A[x]$,! 10 ($\Leftrightarrow E$: 5)	i
$R[x, y] \ \& \ A[x]$,! 11 ($\Rightarrow E$: 9,10)	i
$(R[x, y] \ \& \ A[x])$,! 12 ((I: 11)	i
$\exists z(R[z, y] \ \& \ A[z])$,! 13 ($\exists I$: 12)	i
$((R \upharpoonright A)^I)[y] \Rightarrow \exists z(R[z, y] \ \& \ A[z])$,! 14 ($\Rightarrow I$: 6,13)	i
$\exists z(R[z, y] \ \& \ A[z])$,! 15 (Prem)	i
$(R[x, y] \ \& \ A[x])$,! 16 ($\exists E$: 15)	i
$R[x, y] \ \& \ A[x]$,! 17 ((E: 16)	i
$R[x, y] \ \& \ A[x] \Rightarrow (R \upharpoonright A)[x, y]$,! 18 ($\Leftrightarrow E$: 5)	i
$(R \upharpoonright A)[x, y]$,! 19 ($\Rightarrow E$: 17,18)	i
$\exists x (R \upharpoonright A)[x, y]$,! 20 ($\exists I$: 19)	i
$\exists x (R \upharpoonright A)[x, y] \Rightarrow ((R \upharpoonright A)^I)[y]$,! 21 ($\Leftrightarrow E$: 3)	i
$((R \upharpoonright A)^I)[y]$,! 22 ($\Rightarrow E$: 20,21)	i
$\exists z(R[z, y] \ \& \ A[z]) \Rightarrow ((R \upharpoonright A)^I)[y]$,! 23 ($\Rightarrow I$: 15,22)	i
$((R \upharpoonright A)^I)[y] \Leftrightarrow \exists z(R[z, y] \ \& \ A[z])$,! 24 ($\Leftrightarrow I$: 14,23)	i
$(((R \upharpoonright A)^I)[y] \Leftrightarrow \exists z(R[z, y] \ \& \ A[z]))$,! 25 ((I: 24)	i
$\forall R \forall A \forall y (((R \upharpoonright A)^I)[y] \Leftrightarrow \exists z(R[z, y] \ \& \ A[z]))$! 26 ($\forall I$: 1,25)	i

□

! P47 and P48 are used to translate the multiplication axiom (Mult) into terms involving restrictions and images. i

! 47. i

$\vdash \forall R \forall A \{y : \exists z(R[z,y] \ \& \ A[z])\} \equiv ((R \sqsupset A)^I)$ i

R, A , ! 1 (Prem) i

$\forall x (\{y : \exists z(R[z,y] \ \& \ A[z])\}[x] \Leftrightarrow \exists z(R[z,x] \ \& \ A[z]))$, ! 2 (Pred) i

x , ! 3 (Prem) i

$(\{y : \exists z(R[z,y] \ \& \ A[z])\}[x] \Leftrightarrow \exists z(R[z,x] \ \& \ A[z]))$, ! 4 ($\forall E$: 2) i

$\{y : \exists z(R[z,y] \ \& \ A[z])\}[x] \Leftrightarrow \exists z(R[z,x] \ \& \ A[z])$, ! 5 ((E) : 4) i

$(((R \sqsupset A)^I)[x] \Leftrightarrow \exists z(R[z,x] \ \& \ A[z]))$, ! 6 ($\forall E$: P46) i

$((R \sqsupset A)^I)[x] \Leftrightarrow \exists z(R[z,x] \ \& \ A[z])$, ! 7 ((E) : 6) i

$\{y : \exists z(R[z,y] \ \& \ A[z])\}[x]$, ! 8 (Prem) i

$\{y : \exists z(R[z,y] \ \& \ A[z])\}[x] \Rightarrow \exists z(R[z,x] \ \& \ A[z])$, ! 9 ($\Leftrightarrow E$: 5) i

$\exists z(R[z,x] \ \& \ A[z])$, ! 10 ($\Rightarrow E$: 8,9) i

$\exists z(R[z,x] \ \& \ A[z]) \Rightarrow ((R \sqsupset A)^I)[x]$, ! 11 ($\Leftrightarrow E$: 7) i

$((R \sqsupset A)^I)[x]$, ! 12 ($\Rightarrow E$: 10,11) i

$\{y : \exists z(R[z,y] \ \& \ A[z])\}[x] \Rightarrow ((R \sqsupset A)^I)[x]$, ! 13 ($\Rightarrow I$: 8,12) i

$((R \sqsupset A)^I)[x]$, ! 14 (Prem) i

$((R \sqsupset A)^I)[x] \Rightarrow \exists z(R[z,x] \ \& \ A[z])$, ! 15 ($\Leftrightarrow E$: 7) i

$\exists z(R[z,x] \ \& \ A[z])$, ! 16 ($\Rightarrow E$: 14,15) i

$\exists z(R[z,x] \ \& \ A[z]) \Rightarrow \{y : \exists z(R[z,y] \ \& \ A[z])\}[x]$, ! 17 ($\Leftrightarrow E$: 5) i

$\{y : \exists z(R[z,y] \ \& \ A[z])\}[x]$, ! 18 ($\Rightarrow E$: 16,17) i

$((R \sqsupset A)^I)[x] \Rightarrow \{y : \exists z(R[z,y] \ \& \ A[z])\}[x]$, ! 19 ($\Rightarrow I$: 14,18) i

$\{y : \exists z(R[z,y] \ \& \ A[z])\}[x] \Leftrightarrow ((R \sqsupset A)^I)[x]$, ! 20 ($\Leftrightarrow I$: 13,19) i

$(\{y : \exists z(\mathbf{R}[z,y] \ \& \ \mathbf{A}[z])\}[\mathbf{x}] \Leftrightarrow ((\mathbf{R} \uparrow \mathbf{A})^{\mathbf{I}})[\mathbf{x}])$
, ! 21 ((I: 20) i
 $\forall x (\{y : \exists z(\mathbf{R}[z,y] \ \& \ \mathbf{A}[z])\}[\mathbf{x}] \Leftrightarrow ((\mathbf{R} \uparrow \mathbf{A})^{\mathbf{I}})[\mathbf{x}])$
, ! 22 (\forall I: 3,21) i
 $\{y : \exists z(\mathbf{R}[z,y] \ \& \ \mathbf{A}[z])\} \equiv ((\mathbf{R} \uparrow \mathbf{A})^{\mathbf{I}})$, ! 23 (\mathbb{S} I: II1.1,22) i
 $\forall R \forall A \{y : \exists z(\mathbf{R}[z,y] \ \& \ \mathbf{A}[z])\} \equiv ((\mathbf{R} \uparrow \mathbf{A})^{\mathbf{I}})$! 24 (\forall I: 1,23) i
□

! 48. P48 is proven as a consequence of P47. An alternative approach would be to follow the lines of the proofs for P46 and P47. i

$\vdash \forall R \forall x \{y : \mathbf{R}[x,y]\} \equiv ((\mathbf{R} \uparrow (x^\bullet))^{\mathbf{I}})$ i
 \mathbf{R}, \mathbf{a} , ! 1 (Prem) i
 $\{y : \exists z(\mathbf{R}[z,y] \ \& \ (\mathbf{a}^\bullet)[z])\} \equiv ((\mathbf{R} \uparrow (\mathbf{a}^\bullet))^{\mathbf{I}})$
, ! 2 (\forall E: P47) i
 $\forall x (\{y : \exists z(\mathbf{R}[z,y] \ \& \ (\mathbf{a}^\bullet)[z])\}[\mathbf{x}] \Leftrightarrow \exists z(\mathbf{R}[z,\mathbf{x}] \ \& \ (\mathbf{a}^\bullet)[z]))$
, ! 3 (Pred) i
 $\forall x (\{y : \mathbf{R}[\mathbf{a},y]\}[\mathbf{x}] \Leftrightarrow \mathbf{R}[\mathbf{a},\mathbf{x}])$, ! 4 (Pred) i
 \mathbf{x} , ! 5 (Prem) i
 $(\{y : \exists z(\mathbf{R}[z,y] \ \& \ (\mathbf{a}^\bullet)[z])\}[\mathbf{x}] \Leftrightarrow \exists z(\mathbf{R}[z,\mathbf{x}] \ \& \ (\mathbf{a}^\bullet)[z]))$
, ! 6 (\forall E: 5) i
 $\{y : \exists z(\mathbf{R}[z,y] \ \& \ (\mathbf{a}^\bullet)[z])\}[\mathbf{x}] \Leftrightarrow \exists z(\mathbf{R}[z,\mathbf{x}] \ \& \ (\mathbf{a}^\bullet)[z])$
, ! 7 ((): 6) i
 $(\{y : \mathbf{R}[\mathbf{a},y]\}[\mathbf{x}] \Leftrightarrow \mathbf{R}[\mathbf{a},\mathbf{x}])$, ! 8 (\forall E: 7) i
 $\{y : \mathbf{R}[\mathbf{a},y]\}[\mathbf{x}] \Leftrightarrow \mathbf{R}[\mathbf{a},\mathbf{x}]$, ! 9 ((E: 8) i
 $\{y : \exists z(\mathbf{R}[z,y] \ \& \ (\mathbf{a}^\bullet)[z])\}[\mathbf{x}]$, ! 10 (Prem) i
 $\{y : \exists z(\mathbf{R}[z,y] \ \& \ (\mathbf{a}^\bullet)[z])\}[\mathbf{x}] \Rightarrow \exists z(\mathbf{R}[z,\mathbf{x}] \ \& \ (\mathbf{a}^\bullet)[z])$
, ! 11 (\Leftrightarrow E: 7) i
 $\exists z(\mathbf{R}[z,\mathbf{x}] \ \& \ (\mathbf{a}^\bullet)[z])$, ! 12 (\Rightarrow E: 10,11) i
 $(\mathbf{R}[z,\mathbf{x}] \ \& \ (\mathbf{a}^\bullet)[z])$, ! 13 (\exists E: 12) i
 $\mathbf{R}[z,\mathbf{x}] \ \& \ (\mathbf{a}^\bullet)[z]$, ! 14 ((E: 13) i
 $\mathbf{R}[z,\mathbf{x}]$, ! 15 ($\&$ E: 14) i

$(\mathbf{a}^\bullet)[\mathbf{z}]$,! 16 (&E: 14)	i
$((\mathbf{a}^\bullet)[\mathbf{z}] \Rightarrow \mathbf{z} = \mathbf{a})$,! 17 (\forall E: II8.3)	i
$(\mathbf{a}^\bullet)[\mathbf{z}] \Rightarrow \mathbf{z} = \mathbf{a}$,! 18 (()E: 17)	i
$\mathbf{z} = \mathbf{a}$,! 19 (\Rightarrow E: 16,18)	i
$\mathbf{R}[\mathbf{a},\mathbf{x}]$,! 20 (=E: 15,19)	i
$\mathbf{R}[\mathbf{a},\mathbf{x}] \Rightarrow \{y : \mathbf{R}[\mathbf{a},y]\}[\mathbf{x}]$,! 21 (\Leftrightarrow E: 9)	i
$\{y : \mathbf{R}[\mathbf{a},y]\}[\mathbf{x}]$,! 22 (\Rightarrow E: 20,21)	i
$\{y : \exists z(\mathbf{R}[z,y] \& (\mathbf{a}^\bullet)[z])\}[\mathbf{x}] \Rightarrow \{y : \mathbf{R}[\mathbf{a},y]\}[\mathbf{x}]$,! 23 (\Rightarrow I: 10,22)	i
$\{y : \mathbf{R}[\mathbf{a},y]\}[\mathbf{x}]$,! 24 (Prem)	i
$\{y : \mathbf{R}[\mathbf{a},y]\}[\mathbf{x}] \Rightarrow \mathbf{R}[\mathbf{a},\mathbf{x}]$,! 25 (\Leftrightarrow E: 9)	i
$\mathbf{R}[\mathbf{a},\mathbf{x}]$,! 26 (\Rightarrow E: 24,25)	i
$(\mathbf{a}^\bullet)[\mathbf{a}]$,! 27 (\forall E: II8.5)	i
$\mathbf{R}[\mathbf{a},\mathbf{x}] \& (\mathbf{a}^\bullet)[\mathbf{a}]$,! 28 (&I: 26,27)	i
$(\mathbf{R}[\mathbf{a},\mathbf{x}] \& (\mathbf{a}^\bullet)[\mathbf{a}])$,! 29 (()I: 28)	i
$\exists z(\mathbf{R}[z,\mathbf{x}] \& (\mathbf{a}^\bullet)[z])$,! 30 (\exists I: 29)	i
$\exists z(\mathbf{R}[z,\mathbf{x}] \& (\mathbf{a}^\bullet)[z]) \Rightarrow \{y : \exists z(\mathbf{R}[z,y] \& (\mathbf{a}^\bullet)[z])\}[\mathbf{x}]$,! 31 (\Leftrightarrow E: 7)	i
$\{y : \exists z(\mathbf{R}[z,y] \& (\mathbf{a}^\bullet)[z])\}[\mathbf{x}]$,! 32 (\Rightarrow E: 30,31)	i
$\{y : \mathbf{R}[\mathbf{a},y]\}[\mathbf{x}] \Rightarrow \{y : \exists z(\mathbf{R}[z,y] \& (\mathbf{a}^\bullet)[z])\}[\mathbf{x}]$,! 33 (\Rightarrow I: 24,32)	i
$\{y : \exists z(\mathbf{R}[z,y] \& (\mathbf{a}^\bullet)[z])\}[\mathbf{x}] \Leftrightarrow \{y : \mathbf{R}[\mathbf{a},y]\}[\mathbf{x}]$,! 34 (\Leftrightarrow I: 10,22)	i
$(\{y : \exists z(\mathbf{R}[z,y] \& (\mathbf{a}^\bullet)[z])\}[\mathbf{x}] \Leftrightarrow \{y : \mathbf{R}[\mathbf{a},y]\}[\mathbf{x}])$,! 35 (()I: 34)	i
$\forall x (\{y : \exists z(\mathbf{R}[z,y] \& (\mathbf{a}^\bullet)[z])\}[\mathbf{x}] \Leftrightarrow \{y : \mathbf{R}[\mathbf{a},y]\}[\mathbf{x}])$,! 36 (\forall I: 5,35)	i
$\{y : \exists z(\mathbf{R}[z,y] \& (\mathbf{a}^\bullet)[z])\} \equiv \{y : \mathbf{R}[\mathbf{a},y]\}$,! 37 (\equiv I: III1.7,36)	i

$\{y : \exists z(\mathbf{R}[z,y] \ \& \ (\mathbf{a}^\bullet)[z])\} \equiv \{y : \mathbf{R}[\mathbf{a},y]\}$
 $\& \{y : \exists z(\mathbf{R}[z,y] \ \& \ (\mathbf{a}^\bullet)[z])\} \equiv ((\mathbf{R} \upharpoonright (\mathbf{a}^\bullet))^{\mathbb{I}})$
,! 38 (&I: 2,37) i

$(\{y : \exists z(\mathbf{R}[z,y] \ \& \ (\mathbf{a}^\bullet)[z])\} \equiv \{y : \mathbf{R}[\mathbf{a},y]\})$
 $\& \{y : \exists z(\mathbf{R}[z,y] \ \& \ (\mathbf{a}^\bullet)[z])\} \equiv ((\mathbf{R} \upharpoonright (\mathbf{a}^\bullet))^{\mathbb{I}})$
 $\Rightarrow \{y : \mathbf{R}[\mathbf{a},y]\} \equiv ((\mathbf{R} \upharpoonright (\mathbf{a}^\bullet))^{\mathbb{I}})$
,! 39 (\forall E: III.19) i

$\{y : \exists z(\mathbf{R}[z,y] \ \& \ (\mathbf{a}^\bullet)[z])\} \equiv \{y : \mathbf{R}[\mathbf{a},y]\}$
 $\& \{y : \exists z(\mathbf{R}[z,y] \ \& \ (\mathbf{a}^\bullet)[z])\} \equiv ((\mathbf{R} \upharpoonright (\mathbf{a}^\bullet))^{\mathbb{I}})$
 $\Rightarrow \{y : \mathbf{R}[\mathbf{a},y]\} \equiv ((\mathbf{R} \upharpoonright (\mathbf{a}^\bullet))^{\mathbb{I}})$
,! 40 (()E: 39) i

$\{y : \mathbf{R}[\mathbf{a},y]\} \equiv ((\mathbf{R} \upharpoonright (\mathbf{a}^\bullet))^{\mathbb{I}})$
,! 41 (\Rightarrow E: 38,40) i

$\forall \mathbf{R} \forall \mathbf{x} \{y : \mathbf{R}[\mathbf{x},y]\} \equiv ((\mathbf{R} \upharpoonright (\mathbf{x}^\bullet))^{\mathbb{I}})$
,! 42 (\forall I: 1,41) i

□

! 49. \upharpoonright represents image restriction. i

$\mathbb{D} \upharpoonright ; (\mathbf{R} \upharpoonright \mathbf{B}) ; ; \{a,b : \mathbf{R}[a,b] \ \& \ \mathbf{B}[b]\}$ i

! 50. **Fundamental Proposition of Image Restrictions.** i

$\vdash \forall \mathbf{R} \forall \mathbf{B} \forall \mathbf{x} \forall \mathbf{y} ((\mathbf{R} \upharpoonright \mathbf{B})[\mathbf{x},\mathbf{y}] \Leftrightarrow \mathbf{R}[\mathbf{x},\mathbf{y}] \ \& \ \mathbf{B}[\mathbf{y}])$ i

\mathbf{R}, \mathbf{B} ,! 1 (Prem) i

$\forall \mathbf{x} \forall \mathbf{y} (\{a,b : \mathbf{R}[a,b] \ \& \ \mathbf{B}[b]\}[\mathbf{x},\mathbf{y}] \Leftrightarrow \mathbf{R}[\mathbf{x},\mathbf{y}] \ \& \ \mathbf{B}[\mathbf{y}])$
,! 2 (Pred) i

$\forall \mathbf{x} \forall \mathbf{y} ((\mathbf{R} \upharpoonright \mathbf{B})[\mathbf{x},\mathbf{y}] \Leftrightarrow \mathbf{R}[\mathbf{x},\mathbf{y}] \ \& \ \mathbf{B}[\mathbf{y}])$,! 3 (\mathbb{D} I: P49,2) i

$\forall \mathbf{R} \forall \mathbf{B} \forall \mathbf{x} \forall \mathbf{y} ((\mathbf{R} \upharpoonright \mathbf{B})[\mathbf{x},\mathbf{y}] \Leftrightarrow \mathbf{R}[\mathbf{x},\mathbf{y}] \ \& \ \mathbf{B}[\mathbf{y}])$
! 4 (\forall I: 1,3) i

□

! P51 and P52 establish the duality of image with domain restrictions. i

! 51. i

$\vdash \forall \mathbf{R} \forall \mathbf{B} (\mathbf{R} \upharpoonright \mathbf{B}) \equiv ((\mathbf{R}^*) \upharpoonright (\mathbf{B})^*)$ i

\mathbf{R}, \mathbf{B} ,! 1 (Prem) i

\mathbf{x}, \mathbf{y} ,! 2 (Prem) i

$((\mathbf{R} \upharpoonright \mathbf{B})[\mathbf{x},\mathbf{y}] \Leftrightarrow \mathbf{R}[\mathbf{x},\mathbf{y}] \ \& \ \mathbf{B}[\mathbf{y}])$,! 3 (\forall E: P50) i

$(\mathbf{R} \upharpoonright \mathbf{B})[\mathbf{x},\mathbf{y}] \Leftrightarrow \mathbf{R}[\mathbf{x},\mathbf{y}] \ \& \ \mathbf{B}[\mathbf{y}]$,! 4 (()E: 3) i

$((R^*)[y,x] \Leftrightarrow R[x,y])$,! 5 ($\forall E$: C3.2) ;
 $(R^*)[y,x] \Leftrightarrow R[x,y]$,! 6 ($(\Leftrightarrow)E$: 5) ;
 $(((R^*) \lceil B)[y,x] \Leftrightarrow (R^*)[y,x] \& B[y])$,! 7 ($\forall E$: P2) ;
 $((R^*) \lceil B)[y,x] \Leftrightarrow (R^*)[y,x] \& B[y]$,! 8 ($(\Leftrightarrow)E$: 7) ;
 $((((R^*) \lceil B)^*)[x,y] \Leftrightarrow ((R^*) \lceil B)[y,x])$,! 9 ($\forall E$: C3.2) ;
 $((((R^*) \lceil B)^*)[x,y] \Leftrightarrow ((R^*) \lceil B)[y,x]$,! 10 ($(\Leftrightarrow)E$: 9) ;
 $(R \lfloor B)[x,y]$,! 11 (Prem) ;
 $(R \lfloor B)[x,y] \Rightarrow R[x,y] \& B[y]$,! 12 ($(\Leftrightarrow)E$: 4) ;
 $R[x,y] \& B[y]$,! 13 ($(\Rightarrow)E$: 11,12) ;
 $R[x,y]$,! 14 ($(\&E)$: 13) ;
 $R[x,y] \Rightarrow (R^*)[y,x]$,! 15 ($(\Leftrightarrow)E$: 6) ;
 $(R^*)[y,x]$,! 16 ($(\Rightarrow)E$: 14,15) ;
 $B[y]$,! 17 ($(\&E)$: 13) ;
 $(R^*)[y,x] \& B[y]$,! 18 ($(\&I)$: 16,17) ;
 $(R^*)[y,x] \& B[y] \Rightarrow ((R^*) \lceil B)[y,x]$,! 19 ($(\Leftrightarrow)E$: 8) ;
 $((R^*) \lceil B)[y,x]$,! 20 ($(\Rightarrow)E$: 18,19) ;
 $((R^*) \lceil B)[y,x] \Rightarrow (((R^*) \lceil B)^*)[x,y]$,! 21 ($(\Leftrightarrow)E$: 10) ;
 $((((R^*) \lceil B)^*)[x,y]$,! 22 ($(\Rightarrow)E$: 20,21) ;
 $(R \lfloor B)[x,y] \Rightarrow (((R^*) \lceil B)^*)[x,y]$,! 23 ($(\Rightarrow)I$: 11,22) ;
 $((((R^*) \lceil B)^*)[x,y]$,! 24 (Prem) ;
 $((((R^*) \lceil B)^*)[x,y] \Rightarrow ((R^*) \lceil B)[y,x]$,! 25 ($(\Leftrightarrow)E$: 10) ;
 $((R^*) \lceil B)[y,x]$,! 26 ($(\Rightarrow)E$: 24,25) ;
 $((R^*) \lceil B)[y,x] \Rightarrow (R^*)[y,x] \& B[y]$,! 27 ($(\Leftrightarrow)E$: 8) ;
 $(R^*)[y,x] \& B[y]$,! 28 ($(\Rightarrow)E$: 26,27) ;

$(R^*)[y, x]$, ! 29 (&E: 28)	i
$(R^*)[y, x] \Rightarrow R[x, y]$, ! 30 (\Leftrightarrow E: 6)	i
$R[x, y]$, ! 31 (\Rightarrow E: 29, 30)	i
$B[y]$, ! 32 (&E: 28)	i
$R[x, y] \ \& \ B[y]$, ! 33 (&I: 31, 32)	i
$R[x, y] \ \& \ B[y] \Rightarrow (R \lfloor B)[x, y]$, ! 34 (\Leftrightarrow E: 4)	i
$(R \lfloor B)[x, y]$, ! 35 (\Rightarrow E: 33, 34)	i
$((R^*) \lceil B)^*[x, y] \Rightarrow (R \lfloor B)[x, y]$, ! 36 (\Rightarrow I: 24, 35)	i
$(R \lfloor B)[x, y] \Leftrightarrow ((R^*) \lceil B)^*[x, y]$, ! 37 (\Leftrightarrow I: 23, 36)	i
$((R \lfloor B)[x, y] \Leftrightarrow ((R^*) \lceil B)^*[x, y])$, ! 38 ((I: 37)	i
$\forall x \forall y ((R \lfloor B)[x, y] \Leftrightarrow ((R^*) \lceil B)^*[x, y])$, ! 39 (\forall I: 2, 38)	i
$(R \lfloor B) \equiv ((R^*) \lceil B)^*$, ! 40 (\S I: C1.5)	i
$\forall R \forall B (R \lfloor B) \equiv ((R^*) \lceil B)^*$! 41 (\forall I: 1, 40)	i
\square		
! 52.		i
$\vdash \forall R \forall B ((R \lfloor B)^* \equiv (R^*) \lceil B)$		i
R, B	, ! 1 (Prem)	i
$(R \lfloor B) \equiv ((R^*) \lceil B)^*$, ! 2 (\forall E: P51)	i
$((R \lfloor B) \equiv ((R^*) \lceil B)^*) \Rightarrow ((R \lfloor B)^* \equiv (R^*) \lceil B)$, ! 3 (\forall E: C3.26)	i
$(R \lfloor B) \equiv ((R^*) \lceil B)^* \Rightarrow ((R \lfloor B)^* \equiv (R^*) \lceil B)$, ! 4 ((E: 3)	i
$((R \lfloor B)^* \equiv (R^*) \lceil B)$, ! 5 (\Rightarrow E: 2, 4)	i
$\forall R \forall B ((R \lfloor B)^* \equiv (R^*) \lceil B)$! 6 (\forall I: 1, 5)	i
\square		
! 53.		i
$\vdash \forall R \forall A \forall B (\forall x \forall y (R[x, y] \Rightarrow (A[x] \Leftrightarrow B[y]))$ $\Rightarrow (R \lceil A) \equiv (R \lfloor B))$		i

R, A, B	,! 1 (Prem)	i
$\forall x \forall y (R[x, y] \Rightarrow (A[x] \Leftrightarrow B[y]))$,! 2 (Prem)	i
x, y	,! 3 (Prem)	i
$(R[x, y] \Rightarrow (A[x] \Leftrightarrow B[y]))$,! 4 ($\forall E$: 2)	i
$R[x, y] \Rightarrow (A[x] \Leftrightarrow B[y])$,! 5 ($(\Rightarrow E)$: 4)	i
$((R \lceil A)[x, y] \Leftrightarrow R[x, y] \ \& \ A[x])$,! 6 ($\forall E$: P2)	i
$(R \lceil A)[x, y] \Leftrightarrow R[x, y] \ \& \ A[x]$,! 7 ($(\Rightarrow E)$: 6)	i
$((R \lfloor B)[x, y] \Leftrightarrow R[x, y] \ \& \ B[y])$,! 8 ($\forall E$: P50)	i
$(R \lfloor B)[x, y] \Leftrightarrow R[x, y] \ \& \ B[y]$,! 9 ($(\Rightarrow E)$: 8)	i
$(R \lceil A)[x, y]$,! 10 (Prem)	i
$(R \lceil A)[x, y] \Rightarrow R[x, y] \ \& \ A[x]$,! 11 ($\Leftrightarrow E$: P7)	i
$R[x, y] \ \& \ A[x]$,! 12 ($\Rightarrow E$: 10, 11)	i
$R[x, y]$,! 13 ($\& E$: 12)	i
$A[x]$,! 14 ($\& E$: 12)	i
$(A[x] \Leftrightarrow B[y])$,! 15 ($\Rightarrow E$: 5, 13)	i
$A[x] \Leftrightarrow B[y]$,! 16 ($(\Rightarrow E)$: 15)	i
$A[x] \Rightarrow B[y]$,! 17 ($\Leftrightarrow E$: 16)	i
$B[y]$,! 18 ($\Rightarrow E$: 14, 17)	i
$R[x, y] \ \& \ B[y]$,! 19 ($\& I$: 13, 18)	i
$R[x, y] \ \& \ B[y] \Rightarrow (R \lfloor B)[x, y]$,! 20 ($\Leftrightarrow E$: 9)	i
$(R \lfloor B)[x, y]$,! 21 ($\Rightarrow E$: 19, 20)	i
$(R \lceil A)[x, y] \Rightarrow (R \lfloor B)[x, y]$,! 22 ($\Rightarrow I$: 10, 21)	i
$(R \lfloor B)[x, y]$,! 23 (Prem)	i
$(R \lfloor B)[x, y] \Rightarrow R[x, y] \ \& \ B[y]$,! 24 ($\Leftrightarrow E$: 9)	i
$R[x, y] \ \& \ B[y]$,! 25 ($\Rightarrow E$: 23, 24)	i
$R[x, y]$,! 26 ($\& E$: 25)	i
$B[y]$,! 27 ($\& E$: 25)	i

$(A[x] \leftrightarrow B[y])$,!	28	(\Rightarrow E: 5,26)	i
$A[x] \leftrightarrow B[y]$,!	29	($()$ E: 28)	i
$B[y] \Rightarrow A[x]$,!	30	(\Leftrightarrow E: 29)	i
$A[x]$,!	31	(\Rightarrow E: 27,30)	i
$R[x,y] \ \& \ A[x]$,!	32	($\&$ I: 26,31)	i
$R[x,y] \ \& \ A[x] \Rightarrow (R \lceil A)[x,y]$,!	33	(\Leftrightarrow E: 7)	i
$(R \lceil A)[x,y]$,!	34	(\Rightarrow E: 32,33)	i
$(R \lfloor B)[x,y] \Rightarrow (R \lceil A)[x,y]$,!	35	(\Rightarrow I: 23,34)	i
$(R \lceil A)[x,y] \Leftrightarrow (R \lfloor B)[x,y]$,!	36	(\Leftrightarrow I: 22,35)	i
$((R \lceil A)[x,y] \Leftrightarrow (R \lfloor B)[x,y])$,!	37	($()$ I: 36)	i
$\forall x \forall y ((R \lceil A)[x,y] \Leftrightarrow (R \lfloor B)[x,y])$,!	38	(\forall I: 3,37)	i
$(R \lceil A) \equiv (R \lfloor B)$,!	39	(\equiv I: C1.5,38)	i
$\forall x \forall y (R[x,y] \Rightarrow (A[x] \leftrightarrow B[y])) \Rightarrow (R \lceil A) \equiv (R \lfloor B)$,!	40	(\Rightarrow I: 2,39)	i
$(\forall x \forall y (R[x,y] \Rightarrow (A[x] \leftrightarrow B[y])) \Rightarrow (R \lceil A) \equiv (R \lfloor B))$,!	41	($()$ I: 40)	i
$\forall R \forall A \forall B (\forall x \forall y (R[x,y] \Rightarrow (A[x] \leftrightarrow B[y])) \Rightarrow (R \lceil A) \equiv (R \lfloor B))$!	42	(\forall I: 1,41)	i

□