

**! CHAPTER 1
INCLUSION AND EQUIVALENCE;**

! This chapter introduces the notions of inclusion, of equivalence, and of proper inclusion, of one-place (unary) predicates.

Predicate **P** is included in **Q**, written $P \subseteq Q$, when everything which satisfies **P** also satisfies **Q**.

Predicate **P** is equivalent to **Q** written $P \equiv Q$, when those things which satisfy **P** are exactly those things which satisfy **Q**.

Predicate **P** is properly included in **Q**, written $P \subset Q$, when it is included but not equivalent. i

! 1. \subseteq represents inclusion (of one-place predicates). i

$\S \subseteq ; P \subseteq Q ; \forall x(P[x] \Rightarrow Q[x])$ i

! 2. i

$\vdash \forall P \forall Q \forall x (P[x] \ \& \ P \subseteq Q \Rightarrow Q[x])$ i

P, Q, x , ! 1 (Prem) i

P[x] & P \subseteq Q , ! 2 (Prem) i

P[x] , ! 3 (&E: 2) i

P \subseteq Q , ! 4 (&E: 2) i

$\forall x(P[x] \Rightarrow Q[x])$, ! 5 (\S E: P1,4) i

(P[x] \Rightarrow Q[x]) , ! 6 (\forall E: 5) i

P[x] \Rightarrow Q[x] , ! 7 (()E: 6) i

Q[x] , ! 8 (\Rightarrow E: 3,7) i

P[x] & P \subseteq Q \Rightarrow Q[x] , ! 9 (\Rightarrow I: 2,8) i

(P[x] & P \subseteq Q \Rightarrow Q[x]) , ! 10 (()I: 9) i

$\forall P \forall Q \forall x (P[x] \ \& \ P \subseteq Q \Rightarrow Q[x])$! 11 (\forall I: 1,10) i

□

! 3. i

$\vdash \forall P \forall Q \forall x (\neg Q[x] \ \& \ P \subseteq Q \Rightarrow \neg P[x])$ i

P, Q, x , ! 1 (Prem) i

$\neg Q[x] \ \& \ P \subseteq Q$, ! 2 (Prem) i

$\neg Q[x]$, ! 3 (&E: 2) i

P \subseteq Q , ! 4 (&E: 2) i

$P[x]$,! 5 (Prem)	i
$P[x] \ \& \ P \subseteq Q$,! 6 (&I: 4,5)	i
$(P[x] \ \& \ P \subseteq Q \Rightarrow Q[x])$,! 7 ($\forall E$ P2)	i
$P[x] \ \& \ P \subseteq Q \Rightarrow Q[x]$,! 8 ((E: 7)	i
$Q[x]$,! 9 ($\Rightarrow E$: 6,8)	i
\mathfrak{F}	,! 10 ($\mathfrak{F}I$: 3,9)	i
$P[x] \Rightarrow \mathfrak{F}$,! 11 ($\Rightarrow I$: 5,10)	i
$\neg P[x]$,! 12 ($\neg I$: 11)	i
$\neg Q[x] \ \& \ P \subseteq Q \Rightarrow \neg P[x]$,! 13 ($\Rightarrow I$: 2,12)	i
$(\neg Q[x] \ \& \ P \subseteq Q \Rightarrow \neg P[x])$,! 14 ((I: 13)	i
$\forall P \forall Q \forall x (\neg Q[x] \ \& \ P \subseteq Q \Rightarrow \neg P[x])$! 15 ($\forall I$: 1,14)	i
\square		

! 4. Reflexivity of Inclusion.

$\vdash \forall P P \subseteq P$		i
P	,! 1 (Prem)	i
x	,! 2 (Prem)	i
$P[x]$,! 3 (Prem)	i
$P[x] \Rightarrow P[x]$,! 4 ($\Rightarrow I$: 3,3)	i
$(P[x] \Rightarrow P[x])$,! 5 ((I: 4)	i
$\forall x(P[x] \Rightarrow P[x])$,! 6 ($\forall I$: 2,5)	i
$P \subseteq P$,! 7 ($\mathfrak{S}I$: P1,6)	i
$\forall P P \subseteq P$! 8 ($\forall I$: 1,7)	i
\square		

! 5. Transitivity of Inclusion

$\vdash \forall P \forall Q \forall R (P \subseteq Q \ \& \ Q \subseteq R \Rightarrow P \subseteq R)$		i
P, Q, R	,! 1 (Prem)	i
$P \subseteq Q \ \& \ Q \subseteq R$,! 2 (Prem)	i
$P \subseteq Q$,! 3 (&E: 2)	i

$\forall x(P[x] \Rightarrow Q[x])$,! 4 ($\mathcal{S}E$: P1,3)	i
$Q \subseteq R$,! 5 ($\&E$: 2)	i
$\forall x(Q[x] \Rightarrow R[x])$,! 6 ($\mathcal{S}E$: P1,5)	i
x	,! 7 (Prem)	i
$(P[x] \Rightarrow Q[x])$,! 8 ($\forall E$: 4)	i
$P[x] \Rightarrow Q[x]$,! 9 ($(\Rightarrow)E$: 8)	i
$(Q[x] \Rightarrow R[x])$,! 10 ($\forall E$: 6)	i
$Q[x] \Rightarrow R[x]$,! 11 ($(\Rightarrow)E$: 10)	i
$P[x]$,! 12 (Prem)	i
$Q[x]$,! 13 ($\Rightarrow E$: 9,12)	i
$R[x]$,! 14 ($\Rightarrow E$: 11,13)	i
$P[x] \Rightarrow R[x]$,! 15 ($\Rightarrow I$: 12,14)	i
$(P[x] \Rightarrow R[x])$,! 16 ($(\Rightarrow)I$: 15)	i
$\forall x(P[x] \Rightarrow R[x])$,! 17 ($\forall I$: 7,16)	i
$P \subseteq R$,! 18 ($\mathcal{S}I$: P1,17)	i
$P \subseteq Q \ \& \ Q \subseteq R \Rightarrow P \subseteq R$,! 19 ($\Rightarrow I$: 2,18)	i
$(P \subseteq Q \ \& \ Q \subseteq R \Rightarrow P \subseteq R)$,! 20 ($(\Rightarrow)I$: 19)	i
$\forall P \forall Q \forall R (P \subseteq Q \ \& \ Q \subseteq R \Rightarrow P \subseteq R)$! 21 ($\forall I$: 1,20)	i

□

! 6. 3-Way Transitivity of Inclusion. This easy consequence of transitivity will be useful, since it shortens proofs which would otherwise need to appeal twice to transitivity. i

$\vdash \forall P \forall Q \forall R \forall S (P \subseteq Q \ \& \ Q \subseteq R \ \& \ R \subseteq S \Rightarrow P \subseteq S)$		i
P, Q, R, S	,! 1 (Prem)	i
$P \subseteq Q \ \& \ Q \subseteq R \ \& \ R \subseteq S$,! 2 (Prem)	i
$P \subseteq Q \ \& \ Q \subseteq R$,! 3 ($\&E$: 2)	i
$(P \subseteq Q \ \& \ Q \subseteq R \Rightarrow P \subseteq R)$,! 4 ($\forall E$: P5)	i
$P \subseteq Q \ \& \ Q \subseteq R \Rightarrow P \subseteq R$,! 5 ($(\Rightarrow)E$: 4)	i

$P \subseteq R$, ! 6 ($\Rightarrow E$: 3,5)	i
$R \subseteq S$, ! 7 ($\&E$: 2)	i
$P \subseteq R \ \& \ R \subseteq S$, ! 8 ($\&I$: 6,7)	i
$(P \subseteq R \ \& \ R \subseteq S \Rightarrow P \subseteq S)$, ! 9 ($\forall E$: P5)	i
$P \subseteq R \ \& \ R \subseteq S \Rightarrow P \subseteq S$, ! 10 ($()E$: 9)	i
$P \subseteq S$, ! 11 ($\Rightarrow E$: 8,10)	i
$P \subseteq Q \ \& \ Q \subseteq R \ \& \ R \subseteq S \Rightarrow P \subseteq S$, ! 12 ($\Rightarrow I$: 2,11)	i
$(P \subseteq Q \ \& \ Q \subseteq R \ \& \ R \subseteq S \Rightarrow P \subseteq S)$, ! 13 ($()I$: 12)	i
$\forall P \forall Q \forall R \forall S (P \subseteq Q \ \& \ Q \subseteq R \ \& \ R \subseteq S \Rightarrow P \subseteq S)$, ! 14 ($\forall I$: 1,13)	i

□

! 7. \equiv represents equivalence (of one-place predicates). i

$\mathfrak{E} \equiv ; P \equiv Q ; \forall x(P[x] \leftrightarrow Q[x])$ i

! 8. **Fundamental Proposition of Equivalence, First Half.** i

$\vdash \forall P \forall Q (P \subseteq Q \ \& \ Q \subseteq P \Rightarrow P \equiv Q)$ i

P, Q	, ! 1 (Prem)	i
$P \subseteq Q \ \& \ Q \subseteq P$, ! 2 (Prem)	i
$P \subseteq Q$, ! 3 ($\&E$: 2)	i
$\forall x(P[x] \Rightarrow Q[x])$, ! 4 ($\mathfrak{S}E$: P1,3)	i
$Q \subseteq P$, ! 5 ($\&E$: 2)	i
$\forall x(Q[x] \Rightarrow P[x])$, ! 6 ($\mathfrak{S}E$: P1,5)	i
x	, ! 7 (Prem)	i
$(P[x] \Rightarrow Q[x])$, ! 8 ($\forall E$: 4)	i
$P[x] \Rightarrow Q[x]$, ! 9 ($()E$: 8)	i
$(Q[x] \Rightarrow P[x])$, ! 10 ($\forall E$: 6)	i
$Q[x] \Rightarrow P[x]$, ! 11 ($()E$: 10)	i
$P[x] \leftrightarrow Q[x]$, ! 12 ($\leftrightarrow I$: 9,11)	i

$(P[x] \Leftrightarrow Q[x])$,! 13 ((I: 12)	i
$\forall x(P[x] \Leftrightarrow Q[x])$,! 14 (\forall I: 7,13)	i
$P \equiv Q$,! 15 (\S I: P7,14)	i
$P \subseteq Q \ \& \ Q \subseteq P \Rightarrow P \equiv Q$,! 16 (\Rightarrow I: 2,15)	i
$(P \subseteq Q \ \& \ Q \subseteq P \Rightarrow P \equiv Q)$,! 17 ((I: 16)	i
$\forall P \forall Q (P \subseteq Q \ \& \ Q \subseteq P \Rightarrow P \equiv Q)$! 18 (\forall I: 1,17)	i
\square		

! 9. Reflexivity of Equivalence.

$\vdash \forall P P \equiv P$		i
P	,! 1 (Prem)	i
$P \subseteq P$,! 2 (\forall E: P4)	i
$P \subseteq P \ \& \ P \subseteq P$,! 3 ($\&$ I: 2,2)	i
$(P \subseteq P \ \& \ P \subseteq P \Rightarrow P \equiv P)$,! 4 (\forall E: P8)	i
$P \subseteq P \ \& \ P \subseteq P \Rightarrow P \equiv P$,! 5 ((E: 4)	i
$P \equiv P$,! 6 (\Rightarrow E: 3,5)	i
$\forall P P \equiv P$! 7 (\forall I: 1,6)	i
\square		

! 10. Symmetry of Equivalence.

$\vdash \forall P \forall Q (P \equiv Q \Rightarrow Q \equiv P)$		i
P, Q	,! 1 (Prem)	i
$P \equiv Q$,! 2 (Prem)	i
$\forall x(P[x] \Leftrightarrow Q[x])$,! 3 (\S E: P7,2)	i
x	,! 4 (Prem)	i
$(P[x] \Leftrightarrow Q[x])$,! 5 (\forall E: 3)	i
$P[x] \Leftrightarrow Q[x]$,! 6 ((E: 5)	i
$P[x] \Rightarrow Q[x]$,! 7 (\Leftrightarrow E: 6)	i
$Q[x] \Rightarrow P[x]$,! 8 (\Leftrightarrow E: 6)	i
$Q[x] \Leftrightarrow P[x]$,! 9 (\Leftrightarrow I: 7,8)	i

$(Q[x] \Leftrightarrow P[x])$,! 10 ((I: 9)	i
$\forall x(Q[x] \Leftrightarrow P[x])$,! 11 (\forall I: 4,10)	i
$Q \equiv P$,! 12 (\S I: P7,11)	i
$P \equiv Q \Rightarrow Q \equiv P$,! 13 (\Rightarrow I: 2,12)	i
$(P \equiv Q \Rightarrow Q \equiv P)$,! 14 ((I: 13)	i
$\forall P \forall Q (P \equiv Q \Rightarrow Q \equiv P)$! 15 (\forall I: 1,14)	i

□

! 11. Fundamental Proposition of Equivalence, Second Half, First Part.

$\vdash \forall P \forall Q (P \equiv Q \Rightarrow P \subseteq Q)$		i
P, Q	,! 1 (Prem)	i
$P \equiv Q$,! 2 (Prem)	i
$\forall x(P[x] \Leftrightarrow Q[x])$,! 3 (\S E: P7,2)	i
x	,! 4 (Prem)	i
$(P[x] \Leftrightarrow Q[x])$,! 5 (\forall E: 3)	i
$P[x] \Leftrightarrow Q[x]$,! 6 ((E: 5)	i
$P[x] \Rightarrow Q[x]$,! 7 (\Leftrightarrow E: 6)	i
$(P[x] \Rightarrow Q[x])$,! 8 ((I: 7)	i
$\forall x(P[x] \Rightarrow Q[x])$,! 9 (\forall I: 4,8)	i
$P \subseteq Q$,! 10 (\S I: P1,9)	i
$P \equiv Q \Rightarrow P \subseteq Q$,! 11 (\Rightarrow I: 2,10)	i
$(P \equiv Q \Rightarrow P \subseteq Q)$,! 12 ((I: 11)	i
$\forall P \forall Q (P \equiv Q \Rightarrow P \subseteq Q)$! 13 (\forall I: 1,12)	i

□

! 12. Fundamental Proposition of Equivalence, Second Half, Second Part.

$\vdash \forall P \forall Q (P \equiv Q \Rightarrow Q \subseteq P)$		i
P, Q	,! 1 (Prem)	i

$P \equiv Q$,! 2 (Prem)	i
$(P \equiv Q \Rightarrow Q \equiv P)$,! 3 ($\forall E$: P10)	i
$P \equiv Q \Rightarrow Q \equiv P$,! 4 ($(())E$: 3)	i
$Q \equiv P$,! 5 ($\Rightarrow E$: 2,4)	i
$(Q \equiv P \Rightarrow Q \subseteq P)$,! 6 ($\forall E$: P11)	i
$Q \equiv P \Rightarrow Q \subseteq P$,! 7 ($(())E$: 6)	i
$Q \subseteq P$,! 8 ($\Rightarrow E$: 5,7)	i
$P \equiv Q \Rightarrow Q \subseteq P$,! 9 ($\Rightarrow I$: 2,8)	i
$(P \equiv Q \Rightarrow Q \subseteq P)$,! 10 ($(())I$: 9)	i
$\forall P \forall Q (P \equiv Q \Rightarrow Q \subseteq P)$! 11 ($\forall I$: 1,10)	i

□

! 13. Fundamental Proposition of Equivalence, Second Half.

$\vdash \forall P \forall Q (P \equiv Q \Rightarrow P \subseteq Q \ \& \ Q \subseteq P)$		i
P, Q	,! 1 (Prem)	i
$P \equiv Q$,! 2 (Prem)	i
$(P \equiv Q \Rightarrow P \subseteq Q)$,! 3 ($\forall E$: P11)	i
$P \equiv Q \Rightarrow P \subseteq Q$,! 4 ($(())E$: 3)	i
$P \subseteq Q$,! 5 ($\Rightarrow E$: 2,4)	i
$(P \equiv Q \Rightarrow Q \subseteq P)$,! 6 ($\forall E$: P12)	i
$P \equiv Q \Rightarrow Q \subseteq P$,! 7 ($(())E$: 6)	i
$Q \subseteq P$,! 8 ($\Rightarrow E$: 2,7)	i
$P \subseteq Q \ \& \ Q \subseteq P$,! 9 ($\&I$: 5,8)	i
$P \equiv Q \Rightarrow P \subseteq Q \ \& \ Q \subseteq P$,! 10 ($\Rightarrow I$: 2,9)	i
$(P \equiv Q \Rightarrow P \subseteq Q \ \& \ Q \subseteq P)$,! 11 ($(())I$: 10)	i
$\forall P \forall Q (P \equiv Q \Rightarrow P \subseteq Q \ \& \ Q \subseteq P)$! 12 ($\forall I$: 1,11)	i

□

! 14. Fundamental Proposition of Equivalence.

$\vdash \forall P \forall Q (P \subseteq Q \ \& \ Q \subseteq P \Leftrightarrow P \equiv Q)$		i
P, Q	, ! 1 (Prem)	i
$(P \subseteq Q \ \& \ Q \subseteq P \Rightarrow P \equiv Q)$, ! 2 ($\forall E$: P8)	i
$P \subseteq Q \ \& \ Q \subseteq P \Rightarrow P \equiv Q$, ! 3 ($(\)E$: 2)	i
$(P \equiv Q \Rightarrow P \subseteq Q \ \& \ Q \subseteq P)$, ! 4 ($\forall E$: P13)	i
$P \equiv Q \Rightarrow P \subseteq Q \ \& \ Q \subseteq P$, ! 5 ($(\)E$: 4)	i
$P \subseteq Q \ \& \ Q \subseteq P \Leftrightarrow P \equiv Q$, ! 6 ($\Leftrightarrow I$: 3,5)	i
$(P \subseteq Q \ \& \ Q \subseteq P \Leftrightarrow P \equiv Q)$, ! 7 ($(\)I$: 6)	i
$\forall P \forall Q (P \subseteq Q \ \& \ Q \subseteq P \Leftrightarrow P \equiv Q)$! 8 ($\forall I$: 1,7)	i
\square		

! 15. Transtivity of Equivalence.

$\vdash \forall P \forall Q \forall R (P \equiv Q \ \& \ Q \equiv R \Rightarrow P \equiv R)$		i
P, Q, R	, ! 1 (Prem)	i
$P \equiv Q \ \& \ Q \equiv R$, ! 2 (Prem)	i
$P \equiv Q$, ! 3 ($\&E$)	i
$(P \equiv Q \Rightarrow P \subseteq Q \ \& \ Q \subseteq P)$, ! 4 ($\forall E$: P13)	i
$P \equiv Q \Rightarrow P \subseteq Q \ \& \ Q \subseteq P$, ! 5 ($(\)E$: 4)	i
$P \subseteq Q \ \& \ Q \subseteq P$, ! 6 ($\Rightarrow E$: 3,5)	i
$P \subseteq Q$, ! 7 ($\&E$: 6)	i
$Q \subseteq P$, ! 8 ($\&E$: 6)	i
$Q \equiv R$, ! 9 ($\&E$: 2)	i
$(Q \equiv R \Rightarrow Q \subseteq R \ \& \ R \subseteq Q)$, ! 10 ($\forall E$: P13)	i
$Q \equiv R \Rightarrow Q \subseteq R \ \& \ R \subseteq Q$, ! 11 ($(\)E$: 10)	i
$Q \subseteq R \ \& \ R \subseteq Q$, ! 12 ($\Rightarrow E$: 9,11)	i
$Q \subseteq R$, ! 13 ($\&E$: 12)	i
$R \subseteq Q$, ! 14 ($\&E$: 12)	i
$P \subseteq Q \ \& \ Q \subseteq R$, ! 15 ($\&I$: 7,13)	i

$(P \subseteq Q \ \& \ Q \subseteq R \Rightarrow P \subseteq R)$,! 16 ($\forall E$: P5)	i
$P \subseteq Q \ \& \ Q \subseteq R \Rightarrow P \subseteq R$,! 17 ($()E$: 16)	i
$P \subseteq R$,! 18 ($\Rightarrow E$: 15,17)	i
$R \subseteq Q \ \& \ Q \subseteq P$,! 19 ($\&I$: 8,14)	i
$(R \subseteq Q \ \& \ Q \subseteq P \Rightarrow R \subseteq P)$,! 20 ($\forall E$: P5)	i
$R \subseteq Q \ \& \ Q \subseteq P \Rightarrow R \subseteq P$,! 21 ($()E$: 20)	i
$R \subseteq P$,! 22 ($\Rightarrow E$: 19,21)	i
$P \subseteq R \ \& \ R \subseteq P$,! 23 ($\&I$: 18,22)	i
$(P \subseteq R \ \& \ R \subseteq P \Rightarrow P \equiv R)$,! 24 ($\forall E$: P8)	i
$P \subseteq R \ \& \ R \subseteq P \Rightarrow P \equiv R$,! 25 ($()E$: 24)	i
$P \equiv R$,! 26 ($\Rightarrow E$: 23,25)	i
$P \equiv Q \ \& \ Q \equiv R \Rightarrow P \equiv R$,! 27 ($\Rightarrow I$: 2,26)	i
$(P \equiv Q \ \& \ Q \equiv R \Rightarrow P \equiv R)$,! 28 ($()I$: 27)	i
$\forall P \forall Q \forall R (P \equiv Q \ \& \ Q \equiv R \Rightarrow P \equiv R)$! 29 ($\forall I$: 1,28)	i

□

! P16-P20 present five applications of symmetry to transitivity.

! 16. Applications of Symmetry to Transitivity, n1.

$\vdash \forall P \forall Q \forall R (P \equiv Q \ \& \ Q \equiv R \Rightarrow R \equiv P)$		i
P, Q, R	,! 1 (Prem)	i
$P \equiv Q \ \& \ Q \equiv R$,! 2 (Prem)	i
$(P \equiv Q \ \& \ Q \equiv R \Rightarrow P \equiv R)$,! 3 ($\forall E$: P15)	i
$P \equiv Q \ \& \ Q \equiv R \Rightarrow P \equiv R$,! 4 ($()E$: 3)	i
$P \equiv R$,! 5 ($\Rightarrow E$: 2,4)	i
$(P \equiv R \Rightarrow R \equiv P)$,! 6 ($\forall E$: P10)	i
$P \equiv R \Rightarrow R \equiv P$,! 7 ($()E$: 6)	i
$R \equiv P$,! 8 ($\Rightarrow E$: 5,7)	i

$P \equiv Q \ \& \ Q \equiv R \Rightarrow R \equiv P$,! 9 (\Rightarrow I: 2,8)	i
$(P \equiv Q \ \& \ Q \equiv R \Rightarrow R \equiv P)$,! 10 ($(())$ I: 9)	i
$\forall P \forall Q \forall R (P \equiv Q \ \& \ Q \equiv R \Rightarrow R \equiv P)$! 11 (\forall I: 1,10)	i
\square		

! 17. Applications of Symmetry to Transitivity, n2.

$\vdash \forall P \forall Q \forall R (P \equiv Q \ \& \ R \equiv Q \Rightarrow P \equiv R)$		i
P, Q, R	,! 1 (Prem)	i
$P \equiv Q \ \& \ R \equiv Q$,! 2 (Prem)	i
$P \equiv Q$,! 3 ($\&$ E: 2)	i
$R \equiv Q$,! 4 ($\&$ E: 2)	i
$(R \equiv Q \Rightarrow Q \equiv R)$,! 5 (\forall E: P10)	i
$R \equiv Q \Rightarrow Q \equiv R$,! 6 ($(())$ E: 5)	i
$Q \equiv R$,! 7 (\Rightarrow E: 4,6)	i
$P \equiv Q \ \& \ Q \equiv R$,! 8 ($\&$ I: 3,7)	i
$(P \equiv Q \ \& \ Q \equiv R \Rightarrow P \equiv R)$,! 9 (\forall E: P15)	i
$P \equiv Q \ \& \ Q \equiv R \Rightarrow P \equiv R$,! 10 ($(())$ E: 9)	i
$P \equiv R$,! 11 (\Rightarrow E: 8,10)	i
$P \equiv Q \ \& \ R \equiv Q \Rightarrow P \equiv R$,! 12 (\Rightarrow I: 2,11)	i
$(P \equiv Q \ \& \ R \equiv Q \Rightarrow P \equiv R)$,! 13 ($(())$ I: 12)	i
$\forall P \forall Q \forall R (P \equiv Q \ \& \ R \equiv Q \Rightarrow P \equiv R)$! 14 (\forall I: 1,13)	i
\square		

! 18. Applications of Symmetry to Transitivity, n3.

$\vdash \forall P \forall Q \forall R (P \equiv Q \ \& \ R \equiv Q \Rightarrow R \equiv P)$		i
P, Q, R	,! 1 (Prem)	i
$P \equiv Q \ \& \ R \equiv Q$,! 2 (Prem)	i
$(P \equiv Q \ \& \ R \equiv Q \Rightarrow P \equiv R)$,! 3 (\forall E: P17)	i
$P \equiv Q \ \& \ R \equiv Q \Rightarrow P \equiv R$,! 4 ($(())$ E: 3)	i

$P \equiv R$,! 5 (\Rightarrow E: 2,4)	i
$(P \equiv R \Rightarrow R \equiv P)$,! 6 (\forall E: P10)	i
$P \equiv R \Rightarrow R \equiv P$,! 7 ($($)E: 6)	i
$R \equiv P$,! 8 (\Rightarrow E: 5,7)	i
$P \equiv Q \ \& \ R \equiv Q \Rightarrow R \equiv P$,! 9 (\Rightarrow I: 2,8)	i
$(P \equiv Q \ \& \ R \equiv Q \Rightarrow R \equiv P)$,! 10 ($($)I: 9)	i
$\forall P \forall Q \forall R (P \equiv Q \ \& \ R \equiv Q \Rightarrow R \equiv P)$! 11 (\forall I: 1,10)	i

□

! 19. Applications of Symmetry to Transitivity, n4.

$\vdash \forall P \forall Q \forall R (P \equiv Q \ \& \ P \equiv R \Rightarrow Q \equiv R)$		i
P, Q, R	,! 1 (Prem)	i
$P \equiv Q \ \& \ P \equiv R$,! 2 (Prem)	i
$P \equiv R$,! 3 ($\&$ E: 2)	i
$(P \equiv R \Rightarrow R \equiv P)$,! 4 (\forall E: P10)	i
$P \equiv R \Rightarrow R \equiv P$,! 5 ($($)E: 4)	i
$R \equiv P$,! 6 (\Rightarrow E: 3,5)	i
$P \equiv Q$,! 7 ($\&$ E: 2)	i
$R \equiv P \ \& \ P \equiv Q$,! 8 ($\&$ I: 6,7)	i
$(R \equiv P \ \& \ P \equiv Q \Rightarrow Q \equiv R)$,! 9 (\forall E: P16)	i
$R \equiv P \ \& \ P \equiv Q \Rightarrow Q \equiv R$,! 10 ($($)E: 9)	i
$Q \equiv R$,! 11 (\Rightarrow E: 8,10)	i
$P \equiv Q \ \& \ P \equiv R \Rightarrow Q \equiv R$,! 12 (\Rightarrow I: 2,11)	i
$(P \equiv Q \ \& \ P \equiv R \Rightarrow Q \equiv R)$,! 13 ($($)I: 12)	i
$\forall P \forall Q \forall R (P \equiv Q \ \& \ P \equiv R \Rightarrow Q \equiv R)$! 14 (\forall I: 1,13)	i

□

! 20. Applications of Symmetry to Transitivity, n5.

$\vdash \forall P \forall Q \forall R (P \equiv Q \ \& \ P \equiv R \Rightarrow R \equiv Q)$		i
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P, Q, R	,! 1 (Prem)	i
$P \equiv Q \ \& \ P \equiv R$,! 2 (Prem)	i
$(P \equiv Q \ \& \ P \equiv R \Rightarrow Q \equiv R)$,! 3 ($\forall E$: P19)	i
$P \equiv Q \ \& \ P \equiv R \Rightarrow Q \equiv R$,! 4 ($()E$: 3)	i
$Q \equiv R$,! 5 ($\Rightarrow E$: 2,4)	i
$(Q \equiv R \Rightarrow R \equiv Q)$,! 6 ($\forall E$: P10)	i
$Q \equiv R \Rightarrow R \equiv Q$,! 7 ($()E$: 6)	i
$R \equiv Q$,! 8 ($\Rightarrow E$: 5,7)	i
$P \equiv Q \ \& \ P \equiv R \Rightarrow R \equiv Q$,! 9 ($\Rightarrow I$: 2,8)	i
$(P \equiv Q \ \& \ P \equiv R \Rightarrow R \equiv Q)$,! 10 ($()I$: 9)	i
$\forall P \forall Q \forall R (P \equiv Q \ \& \ P \equiv R \Rightarrow R \equiv Q)$! 11 ($\forall I$: 1,10)	i
\square		

! 21. 3-Way Transitivity of Equivalence. This proposition will save steps in proofs, as two uses of transitivity can be collapsed into one use of 3-way transitivity. i

$\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ Q \equiv R \ \& \ R \equiv S \Rightarrow P \equiv S)$		i
P, Q, R, S	,! 1 (Prem)	i
$P \equiv Q \ \& \ Q \equiv R \ \& \ R \equiv S$,! 2 (Prem)	i
$P \equiv Q \ \& \ Q \equiv R$,! 3 ($\&E$: 2)	i
$(P \equiv Q \ \& \ Q \equiv R \Rightarrow P \equiv R)$,! 4 ($\forall E$: P15)	i
$P \equiv Q \ \& \ Q \equiv R \Rightarrow P \equiv R$,! 5 ($()E$: 4)	i
$P \equiv R$,! 6 ($\Rightarrow E$: 3,5)	i
$R \equiv S$,! 7 ($\&E$: 2)	i
$P \equiv R \ \& \ R \equiv S$,! 8 ($\&I$: 6,7)	i
$(P \equiv R \ \& \ R \equiv S \Rightarrow P \equiv S)$,! 9 ($\forall E$: P15)	i
$P \equiv R \ \& \ R \equiv S \Rightarrow P \equiv S$,! 10 ($()E$: 9)	i
$P \equiv S$,! 11 ($\Rightarrow E$: 8,10)	i
$P \equiv Q \ \& \ Q \equiv R \ \& \ R \equiv S \Rightarrow P \equiv S$,! 12 ($\Rightarrow I$: 2,11)	i

($P \equiv Q \ \& \ Q \equiv R \ \& \ R \equiv S \Rightarrow P \equiv S$) ,! 13 ((I: 12) i

$\forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ Q \equiv R \ \& \ R \equiv S \Rightarrow P \equiv S)$
! 14 (\forall I: 1,13) i

□

! P22-P28 present seven applications of symmetry to 3-way transitivity. i

! 22. Applications of Symmetry to 3-Way Transitivity, n1. i

$\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ Q \equiv R \ \& \ S \equiv R \Rightarrow P \equiv S)$ i

P, Q, R, S ,! 1 (Prem) i

$P \equiv Q \ \& \ Q \equiv R \ \& \ S \equiv R$,! 2 (Prem) i

$P \equiv Q$,! 3 (&E: 2) i

$Q \equiv R$,! 4 (&E: 2) i

$S \equiv R$,! 5 (&E: 2) i

$P \equiv Q \ \& \ Q \equiv R$,! 6 (&I: 3,4) i

($S \equiv R \Rightarrow R \equiv S$) ,! 7 (\forall E: P10) i

$S \equiv R \Rightarrow R \equiv S$,! 8 ((E: 7) i

$R \equiv S$,! 9 (\Rightarrow E: 5,8) i

$P \equiv Q \ \& \ Q \equiv R \ \& \ R \equiv S$,! 10 (&I: 6,9) i

($P \equiv Q \ \& \ Q \equiv R \ \& \ R \equiv S \Rightarrow P \equiv S$) ,! 11 (\forall E: P21) i

$P \equiv Q \ \& \ Q \equiv R \ \& \ R \equiv S \Rightarrow P \equiv S$,! 12 ((E: 11) i

$P \equiv S$,! 13 (\Rightarrow E: 10,12) i

$P \equiv Q \ \& \ Q \equiv R \ \& \ S \equiv R \Rightarrow P \equiv S$,! 14 (\Rightarrow I: 2,13) i

($P \equiv Q \ \& \ Q \equiv R \ \& \ S \equiv R \Rightarrow P \equiv S$) ,! 15 ((I: 14) i

$\forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ Q \equiv R \ \& \ S \equiv R \Rightarrow P \equiv S)$
! 16 (\forall I: 1,15) i

□

! 23. Applications of Symmetry to 3-Way Transitivity, n2. i

$\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ R \equiv Q \ \& \ R \equiv S \Rightarrow P \equiv S)$ i

P, Q, R, S	,! 1 (Prem)	i
$P \equiv Q \ \& \ R \equiv Q \ \& \ R \equiv S$,! 2 (Prem)	i
$P \equiv Q$,! 3 (&E: 2)	i
$R \equiv Q$,! 4 (&E: 2)	i
$R \equiv S$,! 5 (&E: 2)	i
$P \equiv Q \ \& \ R \equiv S$,! 6 (&I: 3,5)	i
$(R \equiv Q \Rightarrow Q \equiv R)$,! 7 (\forall E: P10)	i
$R \equiv Q \Rightarrow Q \equiv R$,! 8 ($(\)$ E: 7)	i
$Q \equiv R$,! 9 (\Rightarrow E: 4,8)	i
$P \equiv Q \ \& \ Q \equiv R \ \& \ R \equiv S$,! 10 (&I: 6,9)	i
$(P \equiv Q \ \& \ Q \equiv R \ \& \ R \equiv S \Rightarrow P \equiv S)$,! 11 (\forall E: P21)	i
$P \equiv Q \ \& \ Q \equiv R \ \& \ R \equiv S \Rightarrow P \equiv S$,! 12 ($(\)$ E: 11)	i
$P \equiv S$,! 13 (\Rightarrow E: 10,12)	i
$P \equiv Q \ \& \ R \equiv Q \ \& \ R \equiv S \Rightarrow P \equiv S$,! 14 (\Rightarrow I: 2, 13)	i
$(P \equiv Q \ \& \ R \equiv Q \ \& \ R \equiv S \Rightarrow P \equiv S)$,! 15 ($(\)$ I: 14)	i
$\forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ R \equiv Q \ \& \ R \equiv S \Rightarrow P \equiv S)$! 16 (\forall I: 1,15)	i

□

! 24. Applications of Symmetry to 3-Way Transitivity, n3.

$\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ R \equiv Q \ \& \ S \equiv R \Rightarrow P \equiv S)$		i
P, Q, R, S	,! 1 (Prem)	i
$P \equiv Q \ \& \ R \equiv Q \ \& \ S \equiv R$,! 2 (Prem)	i
$P \equiv Q$,! 3 (&E: 2)	i
$R \equiv Q$,! 4 (&E: 2)	i
$S \equiv R$,! 5 (&E: 2)	i
$P \equiv Q \ \& \ R \equiv Q$,! 6 (&I: 3,4)	i
$(S \equiv R \Rightarrow R \equiv S)$,! 7 (\forall E: P10)	i

$S \equiv R \Rightarrow R \equiv S$,! 8 (()E: 7)	i
$R \equiv S$,! 9 (\Rightarrow E: 5,8)	i
$P \equiv Q \ \& \ R \equiv Q \ \& \ R \equiv S$,! 10 (&I: 6,9)	i
$(P \equiv Q \ \& \ R \equiv Q \ \& \ R \equiv S \Rightarrow P \equiv S)$,! 11 (\forall E: P23)	i
$P \equiv Q \ \& \ R \equiv Q \ \& \ R \equiv S \Rightarrow P \equiv S$,! 12 (()E: 11)	i
$P \equiv S$,! 13 (\Rightarrow E: 10,12)	i
$P \equiv Q \ \& \ R \equiv Q \ \& \ S \equiv R \Rightarrow P \equiv S$,! 14 (\Rightarrow I: 2,13)	i
$(P \equiv Q \ \& \ R \equiv Q \ \& \ S \equiv R \Rightarrow P \equiv S)$,! 15 (()I: 14)	i
$\forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ R \equiv Q \ \& \ S \equiv R \Rightarrow P \equiv S)$! 16 (\forall I: 1,15)	i

□

! 25. Applications of Symmetry to 3-Way Transitivity, n4.

$\vdash \forall P \forall Q \forall R \forall S (Q \equiv P \ \& \ Q \equiv R \ \& \ R \equiv S \Rightarrow P \equiv S)$		i
P, Q, R, S	,! 1 (Prem)	i
$Q \equiv P \ \& \ Q \equiv R \ \& \ R \equiv S$,! 2 (Prem)	i
$Q \equiv P$,! 3 (&E: 2)	i
$Q \equiv R$,! 4 (&E: 2)	i
$R \equiv S$,! 5 (&E: 2)	i
$Q \equiv R \ \& \ R \equiv S$,! 6 (&I: 4,5)	i
$(Q \equiv P \Rightarrow P \equiv Q)$,! 7 (\forall E: P10)	i
$Q \equiv P \Rightarrow P \equiv Q$,! 8 (()E: 7)	i
$P \equiv Q$,! 9 (\Rightarrow E: 3,8)	i
$P \equiv Q \ \& \ Q \equiv R \ \& \ R \equiv S$,! 10 (&I: 6,9)	i
$(P \equiv Q \ \& \ Q \equiv R \ \& \ R \equiv S \Rightarrow P \equiv S)$,! 11 (\forall E: P21)	i
$P \equiv Q \ \& \ Q \equiv R \ \& \ R \equiv S \Rightarrow P \equiv S$,! 12 (()E: 11)	i
$P \equiv S$,! 13 (\Rightarrow E: 10,12)	i
$Q \equiv P \ \& \ Q \equiv R \ \& \ R \equiv S \Rightarrow P \equiv S$,! 14 (\Rightarrow I: 2,13)	i

$(Q \equiv P \ \& \ Q \equiv R \ \& \ R \equiv S \Rightarrow P \equiv S)$,! 15 (()I: 14) i
 $\forall P \forall Q \forall R \forall S (Q \equiv P \ \& \ Q \equiv R \ \& \ R \equiv S \Rightarrow P \equiv S)$
 ! 16 (\forall I: 1,15) i

□

! 26. Applications of Symmetry to 3-Way Transitivity, n5.

$\vdash \forall P \forall Q \forall R \forall S (Q \equiv P \ \& \ Q \equiv R \ \& \ S \equiv R \Rightarrow P \equiv S)$ i
P, Q, R, S ,! 1 (Prem) i
 $Q \equiv P \ \& \ Q \equiv R \ \& \ S \equiv R$,! 2 (Prem) i
 $Q \equiv P$,! 3 (&E: 2) i
 $Q \equiv R$,! 4 (&E: 2) i
 $S \equiv R$,! 5 (&E: 2) i
 $Q \equiv R \ \& \ S \equiv R$,! 6 (&I: 4,5) i
 $(Q \equiv P \Rightarrow P \equiv Q)$,! 7 (\forall E: P10) i
 $Q \equiv P \Rightarrow P \equiv Q$,! 8 (()E: 7) i
 $P \equiv Q$,! 9 (\Rightarrow E: 3,8) i
 $P \equiv Q \ \& \ Q \equiv R \ \& \ S \equiv R$,! 10 (&I: 6,9) i
 $(P \equiv Q \ \& \ Q \equiv R \ \& \ S \equiv R \Rightarrow P \equiv S)$,! 11 (\forall E: P22) i
 $P \equiv Q \ \& \ Q \equiv R \ \& \ S \equiv R \Rightarrow P \equiv S$,! 12 (()E: 11) i
 $P \equiv S$,! 13 (\Rightarrow E: 10,12) i
 $Q \equiv P \ \& \ Q \equiv R \ \& \ S \equiv R \Rightarrow P \equiv S$,! 14 (\Rightarrow I: 2,13) i
 $(Q \equiv P \ \& \ Q \equiv R \ \& \ S \equiv R \Rightarrow P \equiv S)$,! 15 (()I: 14) i
 $\forall P \forall Q \forall R \forall S (Q \equiv P \ \& \ Q \equiv R \ \& \ S \equiv R \Rightarrow P \equiv S)$
 ! 16 (\forall I: 1,15) i

□

! 27. Applications of Symmetry to 3-Way Transitivity, n6.

$\vdash \forall P \forall Q \forall R \forall S (Q \equiv P \ \& \ R \equiv Q \ \& \ R \equiv S \Rightarrow P \equiv S)$ i
P, Q, R, S ,! 1 (Prem) i
 $Q \equiv P \ \& \ R \equiv Q \ \& \ R \equiv S$,! 2 (Prem) i

$Q \equiv P$,! 3 (&E: 2)	i
$R \equiv Q$,! 4 (&E: 2)	i
$R \equiv S$,! 5 (&E: 2)	i
$R \equiv Q \ \& \ R \equiv S$,! 6 (&I: 5,6)	i
$(Q \equiv P \Rightarrow P \equiv Q)$,! 7 (\forall E: P10)	i
$Q \equiv P \Rightarrow P \equiv Q$,! 8 (()E: 7)	i
$P \equiv Q$,! 9 (\Rightarrow E: 3,8)	i
$P \equiv Q \ \& \ R \equiv Q \ \& \ R \equiv S$,! 10 (&I: 6,9)	i
$(P \equiv Q \ \& \ R \equiv Q \ \& \ R \equiv S \Rightarrow P \equiv S)$,! 11 (\forall E: P23)	i
$P \equiv Q \ \& \ R \equiv Q \ \& \ R \equiv S \Rightarrow P \equiv S$,! 12 (()E: 11)	i
$P \equiv S$,! 13 (\Rightarrow E: 10,12)	i
$Q \equiv P \ \& \ R \equiv Q \ \& \ R \equiv S \Rightarrow P \equiv S$,! 14 (\Rightarrow I: 2,13)	i
$(Q \equiv P \ \& \ R \equiv Q \ \& \ R \equiv S \Rightarrow P \equiv S)$,! 15 (()I: 14)	i
$\forall P \forall Q \forall R \forall S (Q \equiv P \ \& \ R \equiv Q \ \& \ R \equiv S \Rightarrow P \equiv S)$! 16 (\forall I: 1,15)	i

□

! 28. Applications of Symmetry to 3-Way Transitivity, n7.

$\vdash \forall P \forall Q \forall R \forall S (Q \equiv P \ \& \ R \equiv Q \ \& \ S \equiv R \Rightarrow P \equiv S)$		i
P, Q, R, S	,! 1 (Prem)	i
$Q \equiv P \ \& \ R \equiv Q \ \& \ S \equiv R$,! 2 (Prem)	i
$Q \equiv P$,! 3 (&E: 2)	i
$R \equiv Q$,! 4 (&E: 2)	i
$S \equiv R$,! 5 (&E: 2)	i
$R \equiv Q \ \& \ S \equiv R$,! 6 (&I: 4,5)	i
$(Q \equiv P \Rightarrow P \equiv Q)$,! 7 (\forall E: P10)	i
$Q \equiv P \Rightarrow P \equiv Q$,! 8 (()E: 7)	i
$P \equiv Q$,! 9 (\Rightarrow E: 3,8)	i

$P \equiv Q \ \& \ R \equiv Q \ \& \ S \equiv R$,! 10 ($\&I$: 6,9)	i
$(P \equiv Q \ \& \ R \equiv Q \ \& \ S \equiv R \Rightarrow P \equiv S)$,! 11 ($\forall E$: P24)	i
$P \equiv Q \ \& \ R \equiv Q \ \& \ S \equiv R \Rightarrow P \equiv S$,! 12 ($()E$: 11)	i
$P \equiv S$,! 13 ($\Rightarrow E$: 10,12)	i
$Q \equiv P \ \& \ R \equiv Q \ \& \ S \equiv R \Rightarrow P \equiv S$,! 14 ($\Rightarrow I$: 2,13)	i
$(Q \equiv P \ \& \ R \equiv Q \ \& \ S \equiv R \Rightarrow P \equiv S)$,! 15 ($()I$: 14)	i
$\forall P \forall Q \forall R \forall S (Q \equiv P \ \& \ R \equiv Q \ \& \ S \equiv R \Rightarrow P \equiv S)$! 16 ($\forall I$: 1,15)	i

□

! P29-P34 are various versions of the same same: that either the left or right term of an inclusion can be replaced by an equivalent term. i

! 29. i

$\vdash \forall P \forall Q \forall R (P \equiv Q \ \& \ Q \subseteq R \Rightarrow P \subseteq R)$		i
P, Q, R	,! 1 (Prem)	i
$P \equiv Q \ \& \ Q \subseteq R$,! 2 (Prem)	i
$P \equiv Q$,! 3 ($\&E$: 2)	i
$(P \equiv Q \Rightarrow P \subseteq Q)$,! 4 ($\forall E$: P11)	i
$P \equiv Q \Rightarrow P \subseteq Q$,! 5 ($()E$: 4)	i
$P \subseteq Q$,! 6 ($\Rightarrow E$: 3,5)	i
$Q \subseteq R$,! 7 ($\&E$: 2)	i
$P \subseteq Q \ \& \ Q \subseteq R$,! 8 ($\&I$: 6,7)	i
$(P \subseteq Q \ \& \ Q \subseteq R \Rightarrow P \subseteq R)$,! 9 ($\forall E$: P5)	i
$P \subseteq Q \ \& \ Q \subseteq R \Rightarrow P \subseteq R$,! 10 ($()E$: 9)	i
$P \subseteq R$,! 11 ($\Rightarrow E$: 8,10)	i
$P \equiv Q \ \& \ Q \subseteq R \Rightarrow P \subseteq R$,! 12 ($\Rightarrow I$: 2,11)	i
$(P \equiv Q \ \& \ Q \subseteq R \Rightarrow P \subseteq R)$,! 13 ($()I$: 12)	i
$\forall P \forall Q \forall R (P \equiv Q \ \& \ Q \subseteq R \Rightarrow P \subseteq R)$! 14 ($\forall I$: 1,13)	i

□

! 30.

$\vdash \forall P \forall Q \forall R (Q \equiv P \ \& \ Q \subseteq R \Rightarrow P \subseteq R)$

P, Q, R

,! 1 (Prem)

$Q \equiv P \ \& \ Q \subseteq R$

,! 2 (Prem)

$Q \equiv P$

,! 3 (&E: 2)

$(Q \equiv P \Rightarrow P \equiv Q)$

,! 4 (\forall E: P10)

$Q \equiv P \Rightarrow P \equiv Q$

,! 5 (()E: 4)

$P \equiv Q$

,! 6 (\Rightarrow E: 3,5)

$Q \subseteq R$

,! 7 (&E: 2)

$P \equiv Q \ \& \ Q \subseteq R$

,! 8 (&I: 6,7)

$(P \equiv Q \ \& \ Q \subseteq R \Rightarrow P \subseteq R)$

,! 9 (\forall E: P29)

$P \equiv Q \ \& \ Q \subseteq R \Rightarrow P \subseteq R$

,! 10 (()E: 9)

$P \subseteq R$

,! 11 (\Rightarrow E: 8,10)

$Q \equiv P \ \& \ Q \subseteq R \Rightarrow P \subseteq R$

,! 12 (\Rightarrow I: 2,11)

$(Q \equiv P \ \& \ Q \subseteq R \Rightarrow P \subseteq R)$

,! 13 (()I: 12)

$\forall P \forall Q \forall R (Q \equiv P \ \& \ Q \subseteq R \Rightarrow P \subseteq R)$

! 14 (\forall I: 1,13)

□

! 31.

$\vdash \forall P \forall Q \forall R (P \equiv Q \ \& \ R \subseteq Q \Rightarrow R \subseteq P)$

P, Q, R

,! 1 (Prem)

$P \equiv Q \ \& \ R \subseteq Q$

,! 2 (Prem)

$P \equiv Q$

,! 3 (&E: 2)

$(P \equiv Q \Rightarrow Q \subseteq P)$

,! 4 (\forall E: P12)

$P \equiv Q \Rightarrow Q \subseteq P$

,! 5 (()E: 4)

$Q \subseteq P$

,! 6 (\Rightarrow E: 3,5)

$R \subseteq Q$

,! 7 (&E: 2)

$R \subseteq Q \ \& \ Q \subseteq P$

,! 8 (&I: 6,7)

$(R \subseteq Q \ \& \ Q \subseteq P \Rightarrow R \subseteq P)$,! 9 ($\forall E$: P5)	i
$R \subseteq Q \ \& \ Q \subseteq P \Rightarrow R \subseteq P$,! 10 ($()E$: 9)	i
$R \subseteq P$,! 11 ($\Rightarrow E$: 8,10)	i
$P \equiv Q \ \& \ R \subseteq Q \Rightarrow R \subseteq P$,! 12 ($\Rightarrow I$: 2,11)	i
$(P \equiv Q \ \& \ R \subseteq Q \Rightarrow R \subseteq P)$,! 13 ($()I$: 12)	i
$\forall P \forall Q \forall R (P \equiv Q \ \& \ R \subseteq Q \Rightarrow R \subseteq P)$! 14 ($\forall I$: 1,13)	i

□

! 32.

$\vdash \forall P \forall Q \forall R (Q \equiv P \ \& \ R \subseteq Q \Rightarrow R \subseteq P)$		i
P, Q, R	,! 1 (Prem)	i
$Q \equiv P \ \& \ R \subseteq Q$,! 2 (Prem)	i
$Q \equiv P$,! 3 ($\&E$: 2)	i
$(Q \equiv P \Rightarrow P \equiv Q)$,! 4 ($\forall E$: P10)	i
$Q \equiv P \Rightarrow P \equiv Q$,! 5 ($()E$: 4)	i
$P \equiv Q$,! 6 ($\Rightarrow E$: 3,5)	i
$R \subseteq Q$,! 7 ($\&E$: 2)	i
$P \equiv Q \ \& \ R \subseteq Q$,! 8 ($\&I$: 6,7)	i
$(P \equiv Q \ \& \ R \subseteq Q \Rightarrow R \subseteq P)$,! 9 ($\forall E$: P31)	i
$P \equiv Q \ \& \ R \subseteq Q \Rightarrow R \subseteq P$,! 10 ($()E$: 9)	i
$R \subseteq P$,! 11 ($\Rightarrow E$: 8,10)	i
$Q \equiv P \ \& \ R \subseteq Q \Rightarrow R \subseteq P$,! 12 ($\Rightarrow I$: 2,11)	i
$(Q \equiv P \ \& \ R \subseteq Q \Rightarrow R \subseteq P)$,! 13 ($()I$: 12)	i
$\forall P \forall Q \forall R (Q \equiv P \ \& \ R \subseteq Q \Rightarrow R \subseteq P)$! 14 ($\forall I$: 1,13)	i

□

! 33.

$\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ R \equiv S \ \& \ P \subseteq R \Rightarrow Q \subseteq S)$		i
P, Q, R, S	,! 1 (Prem)	i

$P \equiv Q \ \& \ R \equiv S \ \& \ P \subseteq R$, ! 2 (Prem)	i
$P \equiv Q$, ! 3 (&E: 2)	i
$R \equiv S$, ! 4 (&E: 2)	i
$P \subseteq R$, ! 5 (&E: 2)	i
$P \equiv Q \ \& \ P \subseteq R$, ! 6 (&I: 3,5)	i
$(P \equiv Q \ \& \ P \subseteq R \Rightarrow Q \subseteq R)$, ! 7 (\forall E: P30)	i
$P \equiv Q \ \& \ P \subseteq R \Rightarrow Q \subseteq R$, ! 8 (()E: 7)	i
$Q \subseteq R$, ! 9 (\Rightarrow E: 6,8)	i
$R \equiv S \ \& \ Q \subseteq R$, ! 10 (&I: 4,9)	i
$(R \equiv S \ \& \ Q \subseteq R \Rightarrow Q \subseteq S)$, ! 11 (\forall E: P32)	i
$R \equiv S \ \& \ Q \subseteq R \Rightarrow Q \subseteq S$, ! 12 (()E: 11)	i
$Q \subseteq S$, ! 13 (\Rightarrow E: 10,12)	i
$P \equiv Q \ \& \ R \equiv S \ \& \ P \subseteq R \Rightarrow Q \subseteq S$, ! 14 (\Rightarrow I: 2,13)	i
$(P \equiv Q \ \& \ R \equiv S \ \& \ P \subseteq R \Rightarrow Q \subseteq S)$, ! 15 (()I: 14)	i
$\forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ R \equiv S \ \& \ P \subseteq R \Rightarrow Q \subseteq S)$! 16 (\forall I: 1,15)	i

□

! 34. i

$\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ R \equiv S \ \& \ Q \subseteq S \Rightarrow P \subseteq R)$ i

P, Q, R, S	, ! 1 (Prem)	i
$P \equiv Q \ \& \ R \equiv S \ \& \ Q \subseteq S$, ! 2 (Prem)	i
$P \equiv Q$, ! 3 (&E: 2)	i
$R \equiv S$, ! 4 (&E: 2)	i
$Q \subseteq S$, ! 5 (&E: 2)	i
$P \equiv Q \ \& \ Q \subseteq S$, ! 6 (&I: 3,5)	i
$(P \equiv Q \ \& \ Q \subseteq S \Rightarrow P \subseteq S)$, ! 7 (\forall E: P29)	i
$P \equiv Q \ \& \ Q \subseteq S \Rightarrow P \subseteq S$, ! 8 (()E: 7)	i
$P \subseteq S$, ! 9 (\Rightarrow E: 6,8)	i

$R \equiv S \ \& \ P \subseteq S$,! 10 (&I: 4,9)	i
$(R \equiv S \ \& \ P \subseteq S \Rightarrow P \subseteq R)$,! 11 (\forall E: P31)	i
$R \equiv S \ \& \ P \subseteq S \Rightarrow P \subseteq R$,! 12 (()E: 11)	i
$P \subseteq R$,! 13 (\Rightarrow E: 10,12)	i
$P \equiv Q \ \& \ R \equiv S \ \& \ Q \subseteq S \Rightarrow P \subseteq R$,! 14 (\Rightarrow I: 2,13)	i
$(P \equiv Q \ \& \ R \equiv S \ \& \ Q \subseteq S \Rightarrow P \subseteq R)$,! 15 (()I: 14)	i
$\forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ R \equiv S \ \& \ Q \subseteq S \Rightarrow P \subseteq R)$! 16 (\forall I: 1,15)	i

□

! 35.

$\vdash \forall P \forall Q \forall x (P[x] \ \& \ P \equiv Q \Rightarrow Q[x])$		i
P, Q, x	,! 1 (Prem)	i
$P[x] \ \& \ P \equiv Q$,! 2 (Prem)	i
$P[x]$,! 3 (&E: 2)	i
$P \equiv Q$,! 4 (&E: 2)	i
$(P \equiv Q \Rightarrow P \subseteq Q)$,! 5 (\forall E: P11)	i
$P \equiv Q \Rightarrow P \subseteq Q$,! 6 (()E: 5)	i
$P \subseteq Q$,! 7 (\Rightarrow E: 4,6)	i
$P[x] \ \& \ P \subseteq Q$,! 8 (&I: 3,7)	i
$(P[x] \ \& \ P \subseteq Q \Rightarrow Q[x])$,! 9 (\forall E: P2)	i
$P[x] \ \& \ P \subseteq Q \Rightarrow Q[x]$,! 10 (()E: 9)	i
$Q[x]$,! 11 (\Rightarrow E: 8,10)	i
$P[x] \ \& \ P \equiv Q \Rightarrow Q[x]$,! 12 (\Rightarrow I: 2,11)	i
$(P[x] \ \& \ P \equiv Q \Rightarrow Q[x])$,! 13 (()I: 12)	i
$\forall P \forall Q \forall x (P[x] \ \& \ P \equiv Q \Rightarrow Q[x])$! 14 (\forall I: 1,13)	i

□

! 36.

$\vdash \forall P \forall Q \forall x (P[x] \ \& \ Q \equiv P \Rightarrow Q[x])$		i
P, Q, x	,! 1 (Prem)	i
$P[x] \ \& \ Q \equiv P$,! 2 (Prem)	i
$Q \equiv P$,! 3 (&E: 2)	i
$(Q \equiv P \Rightarrow P \equiv Q)$,! 4 (\forall E: P10)	i
$Q \equiv P \Rightarrow P \equiv Q$,! 5 (()E: 4)	i
$P \equiv Q$,! 6 (\Rightarrow E: 3,5)	i
$P[x]$,! 7 (&E: 2)	i
$P[x] \ \& \ P \equiv Q$,! 8 (&I: 6,7)	i
$(P[x] \ \& \ P \equiv Q \Rightarrow Q[x])$,! 9 (\forall E: P35)	i
$P[x] \ \& \ P \equiv Q \Rightarrow Q[x]$,! 10 (()E: 9)	i
$Q[x]$,! 11 (\Rightarrow E: 8,10)	i
$P[x] \ \& \ Q \equiv P \Rightarrow Q[x]$,! 12 (\Rightarrow I: 2,11)	i
$(P[x] \ \& \ Q \equiv P \Rightarrow Q[x])$,! 13 (()I: 12)	i
$\forall P \forall Q \forall x (P[x] \ \& \ Q \equiv P \Rightarrow Q[x])$! 14 (\forall I: 1,13)	i

□

! 37. i

$\vdash \forall P \forall Q \forall x (\neg P[x] \ \& \ P \equiv Q \Rightarrow \neg Q[x])$		i
P, Q, x	,! 1 (Prem)	i
$\neg P[x] \ \& \ P \equiv Q$,! 2 (Prem)	i
$\neg P[x]$,! 3 (&E: 2)	i
$P \equiv Q$,! 4 (&E: 2)	i
$(P \equiv Q \Rightarrow Q \subseteq P)$,! 5 (\forall E: P12)	i
$P \equiv Q \Rightarrow Q \subseteq P$,! 6 (()E: 5)	i
$Q \subseteq P$,! 7 (\Rightarrow E: 4,6)	i
$\neg P[x] \ \& \ Q \subseteq P$,! 8 (&I: 3,7)	i
$(\neg P[x] \ \& \ Q \subseteq P \Rightarrow \neg Q[x])$,! 9 (\forall E: P3)	i
$\neg P[x] \ \& \ Q \subseteq P \Rightarrow \neg Q[x]$,! 10 (()E: 9)	i

$\neg Q[x]$,! 11 ($\Rightarrow E$: 8,10)	i
$\neg P[x] \ \& \ P \equiv Q \Rightarrow \neg Q[x]$,! 12 ($\Rightarrow I$: 2,11)	i
$(\neg P[x] \ \& \ P \equiv Q \Rightarrow \neg Q[x])$,! 13 ($()I$: 12)	i
$\forall P \forall Q \forall x (\neg P[x] \ \& \ P \equiv Q \Rightarrow \neg Q[x])$! 14 ($\forall I$: 1,13)	i
\square		

! 38.

$\vdash \forall P \forall Q \forall x (\neg P[x] \ \& \ Q \equiv P \Rightarrow \neg Q[x])$		i
P, Q, x	,! 1 (Prem)	i
$\neg P[x] \ \& \ Q \equiv P$,! 2 (Prem)	i
$Q \equiv P$,! 3 ($\&E$: 2)	i
$(Q \equiv P \Rightarrow P \equiv Q)$,! 4 ($\forall E$: P10)	i
$Q \equiv P \Rightarrow P \equiv Q$,! 5 ($()E$: 4)	i
$P \equiv Q$,! 6 ($\Rightarrow E$: 3,5)	i
$\neg P[x]$,! 7 ($\&E$: 2)	i
$\neg P[x] \ \& \ P \equiv Q$,! 8 ($\&I$: 6,7)	i
$(\neg P[x] \ \& \ P \equiv Q \Rightarrow \neg Q[x])$,! 9 ($\forall E$: P37)	i
$\neg P[x] \ \& \ P \equiv Q \Rightarrow \neg Q[x]$,! 10 ($()E$: 9)	i
$\neg Q[x]$,! 11 ($\Rightarrow E$: 8,10)	i
$\neg P[x] \ \& \ Q \equiv P \Rightarrow \neg Q[x]$,! 12 ($\Rightarrow I$: 2,11)	i
$(\neg P[x] \ \& \ Q \equiv P \Rightarrow \neg Q[x])$,! 13 ($()I$: 12)	i
$\forall P \forall Q \forall x (\neg P[x] \ \& \ Q \equiv P \Rightarrow \neg Q[x])$! 14 ($\forall I$: 1,13)	i
\square		

! 39. **Symmetry of Non-Equivalence**, the contrapositive of P10.

$\vdash \forall P \forall Q \forall R (\neg P \equiv Q \Rightarrow \neg Q \equiv P)$		i
P, Q	,! 1 (Prem)	i
$\neg P \equiv Q$,! 2 (Prem)	i
$Q \equiv P$,! 3 (Prem)	i

$(Q \equiv P \Rightarrow P \equiv Q)$,! 4 ($\forall E$: P10)	i
$Q \equiv P \Rightarrow P \equiv Q$,! 5 ($(())E$: 4)	i
$P \equiv Q$,! 6 ($\Rightarrow E$: 3,5)	i
\mathfrak{F}	,! 7 ($\mathfrak{F}I$: 2,6)	i
$Q \equiv P \Rightarrow \mathfrak{F}$,! 8 ($\Rightarrow I$: 3,7)	i
$\neg Q \equiv P$,! 9 ($\neg I$: 8)	i
$\neg P \equiv Q \Rightarrow \neg Q \equiv P$,! 10 ($\Rightarrow I$: 2,9)	i
$(\neg P \equiv Q \Rightarrow \neg Q \equiv P)$,! 11 ($(())I$: 10)	i
$\forall P \forall Q \forall R (\neg P \equiv Q \Rightarrow \neg Q \equiv P)$! 12 ($\forall I$: 1,11)	i

□

! P40-P47 are variations on the same theme: terms which are non-equivalent may be replaced by an equivalent term. i

! 40. i

$\vdash \forall P \forall Q \forall R (P \equiv Q \ \& \ \neg Q \equiv R \Rightarrow \neg P \equiv R)$		i
P, Q, R	,! 1 (Prem)	i
$P \equiv Q \ \& \ \neg Q \equiv R$,! 2 (Prem)	i
$P \equiv Q$,! 3 ($\&E$: 2)	i
$\neg Q \equiv R$,! 4 ($\&E$: 2)	i
$P \equiv R$,! 5 (Prem)	i
$P \equiv Q \ \& \ P \equiv R$,! 6 ($\&I$: 3,5)	i
$(P \equiv Q \ \& \ P \equiv R \Rightarrow Q \equiv R)$,! 7 ($\forall E$: P19)	i
$P \equiv Q \ \& \ P \equiv R \Rightarrow Q \equiv R$,! 8 ($(())E$: 7)	i
$Q \equiv R$,! 9 ($\Rightarrow E$: 6,8)	i
\mathfrak{F}	,! 10 ($\mathfrak{F}I$: 4,9)	i
$P \equiv R \Rightarrow \mathfrak{F}$,! 11 ($\Rightarrow I$: 5,10)	i
$\neg P \equiv R$,! 12 ($\neg I$: 11)	i
$P \equiv Q \ \& \ \neg Q \equiv R \Rightarrow \neg P \equiv R$,! 13 ($\Rightarrow I$: 2,12)	i
$(P \equiv Q \ \& \ \neg Q \equiv R \Rightarrow \neg P \equiv R)$,! 14 ($(())I$: 13)	i

$\forall P \forall Q \forall R (P \equiv Q \ \& \ \neg Q \equiv R \Rightarrow \neg P \equiv R)$! 15 ($\forall I$: 1,14) i

□

! 41. i

$\vdash \forall P \forall Q \forall R (P \equiv Q \ \& \ \neg Q \equiv R \Rightarrow \neg R \equiv P)$ i

P, Q, R ,! 1 (Prem) i

$P \equiv Q \ \& \ \neg Q \equiv R$,! 2 (Prem) i

$(P \equiv Q \ \& \ \neg Q \equiv R \Rightarrow \neg P \equiv R)$,! 3 ($\forall E$: P40) i

$P \equiv Q \ \& \ \neg Q \equiv R \Rightarrow \neg P \equiv R$,! 4 ($(\Rightarrow)E$: 3) i

$\neg P \equiv R$,! 5 ($\Rightarrow E$: 2,4) i

$(\neg P \equiv R \Rightarrow \neg R \equiv P)$,! 6 ($\forall E$: P39) i

$\neg P \equiv R \Rightarrow \neg R \equiv P$,! 7 ($(\Rightarrow)E$: 6) i

$\neg R \equiv P$,! 8 ($\Rightarrow E$: 5,7) i

$P \equiv Q \ \& \ \neg Q \equiv R \Rightarrow \neg R \equiv P$,! 9 ($\Rightarrow I$: 2,8) i

$(P \equiv Q \ \& \ \neg Q \equiv R \Rightarrow \neg R \equiv P)$,! 10 ($(\Rightarrow)I$: 9) i

$\forall P \forall Q \forall R (P \equiv Q \ \& \ \neg Q \equiv R \Rightarrow \neg R \equiv P)$! 11 ($\forall I$: 1,10) i

□

! 42. i

$\vdash \forall P \forall Q \forall R (Q \equiv P \ \& \ \neg Q \equiv R \Rightarrow \neg P \equiv R)$ i

P, Q, R ,! 1 (Prem) i

$Q \equiv P \ \& \ \neg Q \equiv R$,! 2 (Prem) i

$Q \equiv P$,! 3 ($\&E$: 2) i

$\neg Q \equiv R$,! 4 ($\&E$: 2) i

$(Q \equiv P \Rightarrow P \equiv Q)$,! 5 ($\forall E$: P10) i

$Q \equiv P \Rightarrow P \equiv Q$,! 6 ($(\Rightarrow)E$: 5) i

$P \equiv Q$,! 7 ($\Rightarrow E$: 3,6) i

$P \equiv Q \ \& \ \neg Q \equiv R$,! 8 ($\&I$: 4,7) i

$(P \equiv Q \ \& \ \neg Q \equiv R \Rightarrow \neg P \equiv R)$,! 9 ($\forall E$: P40) i

$P \equiv Q \ \& \ \neg Q \equiv R \Rightarrow \neg P \equiv R$, ! 10 (() E : 9)	i
$\neg P \equiv R$, ! 11 (\Rightarrow E : 8, 10)	i
$Q \equiv P \ \& \ \neg Q \equiv R \Rightarrow \neg P \equiv R$, ! 12 (\Rightarrow I : 2, 11)	i
$(Q \equiv P \ \& \ \neg Q \equiv R \Rightarrow \neg P \equiv R)$, ! 13 (() I : 12)	i
$\forall P \forall Q \forall R (Q \equiv P \ \& \ \neg Q \equiv R \Rightarrow \neg P \equiv R)$! 14 (\forall I : 1, 13)	i
\square		

! 43.

$\vdash \forall P \forall Q \forall R (Q \equiv P \ \& \ \neg Q \equiv R \Rightarrow \neg R \equiv P)$		i
P, Q, R	, ! 1 (Prem)	i
$Q \equiv P \ \& \ \neg Q \equiv R$, ! 2 (Prem)	i
$(Q \equiv P \ \& \ \neg Q \equiv R \Rightarrow \neg P \equiv R)$, ! 3 (\forall E : P42)	i
$Q \equiv P \ \& \ \neg Q \equiv R \Rightarrow \neg P \equiv R$, ! 4 (() E : 3)	i
$\neg P \equiv R$, ! 5 (\Rightarrow E : 2, 4)	i
$(\neg P \equiv R \Rightarrow \neg R \equiv P)$, ! 6 (\forall E : P39)	i
$\neg P \equiv R \Rightarrow \neg R \equiv P$, ! 7 (() E : 6)	i
$\neg R \equiv P$, ! 8 (\Rightarrow E : 5, 7)	i
$Q \equiv P \ \& \ \neg Q \equiv R \Rightarrow \neg R \equiv P$, ! 9 (\Rightarrow I : 2, 8)	i
$(Q \equiv P \ \& \ \neg Q \equiv R \Rightarrow \neg R \equiv P)$, ! 10 (() I : 9)	i
$\forall P \forall Q \forall R (Q \equiv P \ \& \ \neg Q \equiv R \Rightarrow \neg R \equiv P)$! 11 (\forall I : 1, 10)	i
\square		

! 44.

$\vdash \forall P \forall Q \forall R (P \equiv Q \ \& \ \neg R \equiv Q \Rightarrow \neg P \equiv R)$		i
P, Q, R	, ! 1 (Prem)	i
$P \equiv Q \ \& \ \neg R \equiv Q$, ! 2 (Prem)	i
$P \equiv Q$, ! 3 ($\&$ E : 2)	i
$\neg R \equiv Q$, ! 4 ($\&$ E : 2)	i
$(\neg R \equiv Q \Rightarrow \neg Q \equiv R)$, ! 5 (\forall E : P39)	i
$\neg R \equiv Q \Rightarrow \neg Q \equiv R$, ! 6 (() E : 5)	i

$\neg Q \equiv R$, ! 7 (\Rightarrow E: 4,6)	i
$P \equiv Q \ \& \ \neg Q \equiv R$, ! 8 ($\&$ I: 3,7)	i
$(P \equiv Q \ \& \ \neg Q \equiv R \Rightarrow \neg P \equiv R)$, ! 9 (\forall E: P40)	i
$P \equiv Q \ \& \ \neg Q \equiv R \Rightarrow \neg P \equiv R$, ! 10 ($()$ E: 9)	i
$\neg P \equiv R$, ! 11 (\Rightarrow E: 8,10)	i
$P \equiv Q \ \& \ \neg R \equiv Q \Rightarrow \neg P \equiv R$, ! 12 (\Rightarrow I: 2,11)	i
$(P \equiv Q \ \& \ \neg R \equiv Q \Rightarrow \neg P \equiv R)$, ! 13 ($()$ I: 12)	i
$\forall P \forall Q \forall R (P \equiv Q \ \& \ \neg R \equiv Q \Rightarrow \neg P \equiv R)$! 14 (\forall I: 1,13)	i

□

! 45.

$\vdash \forall P \forall Q \forall R (P \equiv Q \ \& \ \neg R \equiv Q \Rightarrow \neg R \equiv P)$		i
P, Q, R	, ! 1 (Prem)	i
$P \equiv Q \ \& \ \neg R \equiv Q$, ! 2 (Prem)	i
$(P \equiv Q \ \& \ \neg R \equiv Q \Rightarrow \neg P \equiv R)$, ! 3 (\forall E: P44)	i
$P \equiv Q \ \& \ \neg R \equiv Q \Rightarrow \neg P \equiv R$, ! 4 ($()$ E: 3)	i
$\neg P \equiv R$, ! 5 (\Rightarrow E: 2,4)	i
$(\neg P \equiv R \Rightarrow \neg R \equiv P)$, ! 6 (\forall E: P39)	i
$\neg P \equiv R \Rightarrow \neg R \equiv P$, ! 7 ($()$ E: 6)	i
$\neg R \equiv P$, ! 8 (\Rightarrow E: 5,7)	i
$P \equiv Q \ \& \ \neg R \equiv Q \Rightarrow \neg R \equiv P$, ! 9 (\Rightarrow I: 2,8)	i
$(P \equiv Q \ \& \ \neg R \equiv Q \Rightarrow \neg R \equiv P)$, ! 10 ($()$ I: 9)	i
$\forall P \forall Q \forall R (P \equiv Q \ \& \ \neg R \equiv Q \Rightarrow \neg R \equiv P)$! 11 (\forall I: 1,10)	i

□

! 46.

$\vdash \forall P \forall Q \forall R (Q \equiv P \ \& \ \neg R \equiv Q \Rightarrow \neg P \equiv R)$		i
P, Q, R	, ! 1 (Prem)	i
$Q \equiv P \ \& \ \neg R \equiv Q$, ! 2 (Prem)	i

$Q \equiv P$,! 3 ($\&E$: 2)	i
$\neg R \equiv Q$,! 4 ($\&E$: 2)	i
$(Q \equiv P \Rightarrow P \equiv Q)$,! 5 ($\forall E$: P10)	i
$Q \equiv P \Rightarrow P \equiv Q$,! 6 ($(\)E$: 5)	i
$P \equiv Q$,! 7 ($\Rightarrow E$: 3,6)	i
$P \equiv Q \ \& \ \neg R \equiv Q$,! 8 ($\&I$: 4,7)	i
$(P \equiv Q \ \& \ \neg R \equiv Q \Rightarrow \neg P \equiv R)$,! 9 ($\forall E$: P44)	i
$P \equiv Q \ \& \ \neg R \equiv Q \Rightarrow \neg P \equiv R$,! 10 ($(\)E$: 9)	i
$\neg P \equiv R$,! 11 ($\Rightarrow E$: 8,10)	i
$Q \equiv P \ \& \ \neg R \equiv Q \Rightarrow \neg P \equiv R$,! 12 ($\Rightarrow I$: 2,11)	i
$(Q \equiv P \ \& \ \neg R \equiv Q \Rightarrow \neg P \equiv R)$,! 13 ($(\)I$: 12)	i
$\forall P \forall Q \forall R (Q \equiv P \ \& \ \neg R \equiv Q \Rightarrow \neg P \equiv R)$! 14 ($\forall I$: 1,13)	i

□

! 47.

$\vdash \forall P \forall Q \forall R (Q \equiv P \ \& \ \neg R \equiv Q \Rightarrow \neg R \equiv P)$		i
P, Q, R	,! 1 (Prem)	i
$Q \equiv P \ \& \ \neg R \equiv Q$,! 2 (Prem)	i
$(Q \equiv P \ \& \ \neg R \equiv Q \Rightarrow \neg P \equiv R)$,! 3 ($\forall E$: P46)	i
$Q \equiv P \ \& \ \neg R \equiv Q \Rightarrow \neg P \equiv R$,! 4 ($(\)E$: 3)	i
$\neg P \equiv R$,! 5 ($\Rightarrow E$: 2,4)	i
$(\neg P \equiv R \Rightarrow \neg R \equiv P)$,! 6 ($\forall E$: P39)	i
$\neg P \equiv R \Rightarrow \neg R \equiv P$,! 7 ($(\)E$: 6)	i
$\neg R \equiv P$,! 8 ($\Rightarrow E$: 5,7)	i
$Q \equiv P \ \& \ \neg R \equiv Q \Rightarrow \neg R \equiv P$,! 9 ($\Rightarrow I$: 2,8)	i
$(Q \equiv P \ \& \ \neg R \equiv Q \Rightarrow \neg R \equiv P)$,! 10 ($(\)I$: 9)	i
$\forall P \forall Q \forall R (Q \equiv P \ \& \ \neg R \equiv Q \Rightarrow \neg R \equiv P)$! 11 ($\forall I$: 1,10)	i

□

! 48. The Law of The Excluded Middle for Equivalence. i

$\vdash \forall P \forall Q (P \equiv Q \vee \neg P \equiv Q)$ i

P, Q ,! 1 (Prem) i

$\neg (P \equiv Q \vee \neg P \equiv Q)$,! 2 (Prem) i

$P \equiv Q$,! 3 (Prem) i

$P \equiv Q \vee \neg P \equiv Q$,! 4 (\vee I: 3) i

$(P \equiv Q \vee \neg P \equiv Q)$,! 5 ($(\)$ I: 4) i

\mathfrak{F} ,! 6 (\mathfrak{F} I: 2,5) i

$P \equiv Q \Rightarrow \mathfrak{F}$,! 7 (\Rightarrow I: 3,6) i

$\neg P \equiv Q$,! 8 (\neg I: 7) i

$P \equiv Q \vee \neg P \equiv Q$,! 9 (\vee I: 8) i

$(P \equiv Q \vee \neg P \equiv Q)$,! 10 ($(\)$ I: 9) i

\mathfrak{F} ,! 11 (\mathfrak{F} I: 2,10) i

$\neg (P \equiv Q \vee \neg P \equiv Q) \Rightarrow \mathfrak{F}$,! 12 (\Rightarrow I: 2,11) i

$\neg\neg (P \equiv Q \vee \neg P \equiv Q)$,! 13 (\neg I: 12) i

$(P \equiv Q \vee \neg P \equiv Q)$,! 14 (\neg E: 13) i

$\forall P \forall Q (P \equiv Q \vee \neg P \equiv Q)$! 15 (\forall I: 1,14) i

□

! 49. \subset represents proper inclusion (of one-place predicates). i

$\S \subset ; P \subset Q ; P \subseteq Q \ \& \ \neg P \equiv Q$ i

! The development of \subset will be scant, since it will have fewer applications. One particular of importance to note is in the definition of a hereditary predicate in IV.6. i

! 50. Proper Inclusion Is Inclusive i

$\vdash \forall P \forall Q (P \subset Q \Rightarrow P \subseteq Q)$ i

P, Q ,! 1 (Prem) i

$P \subset Q$,! 2 (Prem) i

$P \subseteq Q \ \& \ \neg P \equiv Q$,! 3 (\S E: P49,2) i

$P \subseteq Q$,! 4 (&E: 3)	i
$P \subset Q \Rightarrow P \subseteq Q$,! 5 (\Rightarrow I: 2,4)	i
$(P \subset Q \Rightarrow P \subseteq Q)$,! 6 (()I: 5)	i
$\forall P \forall Q (P \subset Q \Rightarrow P \subseteq Q)$! 7 (\forall I: 1,6)	i
\square		

! 51. Proper Inclusion is Proper.

$\vdash \forall P \forall Q (P \subset Q \Rightarrow \neg P \equiv Q)$		i
P, Q	,! 1 (Prem)	i
$P \subset Q$,! 2 (Prem)	i
$P \subseteq Q \ \& \ \neg P \equiv Q$,! 3 (\S E: P49,2)	i
$\neg P \equiv Q$,! 4 (&E: 3)	i
$P \subset Q \Rightarrow \neg P \equiv Q$,! 5 (\Rightarrow I: 2,4)	i
$(P \subset Q \Rightarrow \neg P \equiv Q)$,! 6 (()I: 5)	i
$\forall P \forall Q (P \subset Q \Rightarrow \neg P \equiv Q)$! 7 (\forall I: 1,6)	i
\square		

! 52. Corollary (Contrapositive) to P51.

$\vdash \forall P \forall Q (P \equiv Q \Rightarrow \neg P \subset Q)$		i
P, Q	,! 1 (Prem)	i
$P \equiv Q$,! 2 (Prem)	i
$P \subset Q$,! 3 (Prem)	i
$(P \subset Q \Rightarrow \neg P \equiv Q)$,! 4 (\forall E: P51)	i
$P \subset Q \Rightarrow \neg P \equiv Q$,! 5 (()E: 4)	i
$\neg P \equiv Q$,! 6 (\Rightarrow E: 3,5)	i
\mathfrak{F}	,! 7 (\mathfrak{F} I: 2,6)	i
$P \subset Q \Rightarrow \mathfrak{F}$,! 8 (\Rightarrow I: 3,7)	i
$\neg P \subset Q$,! 9 (\neg I: 8)	i
$P \equiv Q \Rightarrow \neg P \subset Q$,! 10 (\Rightarrow I: 2,9)	i

$(P \equiv Q \Rightarrow \neg P \subset Q)$,! 11 ((I: 10) i
 $\forall P \forall Q (P \equiv Q \Rightarrow \neg P \subset Q)$! 12 (\forall I: 1,11) i
 \square

! 53. Fundamental Proposition of Proper Inclusion, First Half. i

$\vdash \forall P \forall Q (P \subseteq Q \Rightarrow P \equiv Q \vee P \subset Q)$ i
 P, Q ,! 1 (Prem) i
 $P \subseteq Q$,! 2 (Prem) i
 $(P \equiv Q \vee \neg P \equiv Q)$,! 3 (\forall E: P48) i
 $P \equiv Q \vee \neg P \equiv Q$,! 4 ((E: 3) i
 $P \equiv Q$,! 5 (Prem) i
 $P \equiv Q \vee P \subset Q$,! 6 (\vee I: 5) i
 $P \equiv Q \Rightarrow P \equiv Q \vee P \subset Q$,! 7 (\Rightarrow I: 5,6) i
 $\neg P \equiv Q$,! 8 (Prem) i
 $P \subseteq Q \ \& \ \neg P \equiv Q$,! 9 (&I: 2,8) i
 $P \subset Q$,! 10 (\S I: P49,9) i
 $P \equiv Q \vee P \subset Q$,! 11 (\vee I: 10) i
 $\neg P \equiv Q \Rightarrow P \equiv Q \vee P \subset Q$,! 12 (\Rightarrow I: 8,11) i
 $P \equiv Q \vee P \subset Q$,! 13 (\vee E: 4,7,12) i
 $P \subseteq Q \Rightarrow P \equiv Q \vee P \subset Q$,! 14 (\Rightarrow I: 2,13) i
 $(P \subseteq Q \Rightarrow P \equiv Q \vee P \subset Q)$,! 15 ((I: 14) i
 $\forall P \forall Q (P \subseteq Q \Rightarrow P \equiv Q \vee P \subset Q)$! 16 (\forall I: 1,15) i

\square

! 54. Fundamental Proposition of Proper Inclusion, Second Half. i

$\vdash \forall P \forall Q (P \equiv Q \vee P \subset Q \Rightarrow P \subseteq Q)$ i
 P, Q ,! 1 (Prem) i
 $P \equiv Q \vee P \subset Q$,! 2 (Prem) i
 $P \equiv Q$,! 3 (Prem) i

$(P \equiv Q \Rightarrow P \subseteq Q)$,! 4 ($\forall E$: P11)	i
$P \equiv Q \Rightarrow P \subseteq Q$,! 5 ($(())E$: 4)	i
$P \subseteq Q$,! 6 ($\Rightarrow E$: 3,5)	i
$P \equiv Q \Rightarrow P \subseteq Q$,! 7 ($\Rightarrow I$: 3,6)	i
$P \subset Q$,! 8 (Prem)	i
$(P \subset Q \Rightarrow P \subseteq Q)$,! 9 ($\forall E$: P50)	i
$P \subset Q \Rightarrow P \subseteq Q$,! 10 ($(())E$: 9)	i
$P \subseteq Q$,! 11 ($\Rightarrow E$: 8,10)	i
$P \subset Q \Rightarrow P \subseteq Q$,! 12 ($\Rightarrow I$: 8,11)	i
$P \subseteq Q$,! 13 ($\forall E$: 2,7,12)	i
$P \equiv Q \vee P \subset Q \Rightarrow P \subseteq Q$,! 14 ($\Rightarrow I$: 2,13)	i
$(P \equiv Q \vee P \subset Q \Rightarrow P \subseteq Q)$,! 15 ($(())I$: 14)	i
$\forall P \forall Q (P \equiv Q \vee P \subset Q \Rightarrow P \subseteq Q)$! 16 ($\forall I$: 1,15)	i

□

! 55. Fundamental Proposition of Proper Inclusion.

$\vdash \forall P \forall Q (P \subseteq Q \Leftrightarrow P \equiv Q \vee P \subset Q)$		
P, Q	,! 1 (Prem)	i
$(P \subseteq Q \Rightarrow P \equiv Q \vee P \subset Q)$,! 2 ($\forall E$: P53)	i
$P \subseteq Q \Rightarrow P \equiv Q \vee P \subset Q$,! 3 ($(())E$: 2)	i
$(P \equiv Q \vee P \subset Q \Rightarrow P \subseteq Q)$,! 4 ($\forall E$: P54)	i
$P \equiv Q \vee P \subset Q \Rightarrow P \subseteq Q$,! 5 ($(())E$: 4)	i
$P \subseteq Q \Leftrightarrow P \equiv Q \vee P \subset Q$,! 6 ($\Leftrightarrow E$: 3,5)	i
$(P \subseteq Q \Leftrightarrow P \equiv Q \vee P \subset Q)$,! 7 ($(())I$: 6)	i
$\forall P \forall Q (P \subseteq Q \Leftrightarrow P \equiv Q \vee P \subset Q)$! 8 ($\forall I$: 1,7)	i

□

! 56. Irreflexivity of Proper Inclusion.

$\vdash \forall P \neg P \subset P$		
-------------------------------------	--	--

P	,! 1 (Prem)	i
$P \subset P$,! 2 (Prem)	i
$(P \subset P \Rightarrow \neg P \equiv P)$,! 3 ($\forall E$: P51)	i
$P \subset P \Rightarrow \neg P \equiv P$,! 4 ($(())E$: 3)	i
$\neg P \equiv P$,! 5 ($\Rightarrow E$: 2,4)	i
$P \equiv P$,! 6 ($\forall E$: P9)	i
\mathfrak{F}	,! 7 ($\mathfrak{F}I$: 5,6)	i
$P \subset P \Rightarrow \mathfrak{F}$,! 8 ($\Rightarrow I$: 2,7)	i
$\neg P \subset P$,! 9 ($\neg I$: 8)	i
$\forall P \neg P \subset P$! 10 ($\forall I$: 1,9)	i

□

! 57. Asymmetry of Proper Inclusion.

$\vdash \forall P \forall Q (P \subset Q \Rightarrow \neg Q \subset P)$		i
P, Q	,! 1 (Prem)	i
$P \subset Q$,! 2 (Prem)	i
$(P \subset Q \Rightarrow P \subseteq Q)$,! 3 ($\forall E$: P50)	i
$P \subset Q \Rightarrow P \subseteq Q$,! 4 ($(())E$: 3)	i
$P \subseteq Q$,! 5 ($\Rightarrow E$: 2,4)	i
$Q \subset P$,! 6 (Prem)	i
$Q \subset P \Rightarrow Q \subseteq P$,! 7 ($\forall E$: P50)	i
$Q \subseteq P$,! 8 ($\Rightarrow E$: 6,7)	i
$P \subseteq Q \ \& \ Q \subseteq P$,! 9 ($\&I$: 5,9)	i
$(P \subseteq Q \ \& \ Q \subseteq P \Rightarrow P \equiv Q)$,! 10 ($\forall E$: P8)	i
$P \subseteq Q \ \& \ Q \subseteq P \Rightarrow P \equiv Q$,! 11 ($(())E$: 10)	i
$P \equiv Q$,! 12 ($\Rightarrow E$: 9,11)	i
$(P \subset Q \Rightarrow \neg P \equiv Q)$,! 13 ($\forall E$: P51)	i
$P \subset Q \Rightarrow \neg P \equiv Q$,! 14 ($(())E$: 13)	i

$\neg P \equiv Q$,! 15 (\Rightarrow E: 2,14)	i
\mathfrak{F}	,! 16 (\mathfrak{F} I: 12,15)	i
$Q \subset P \Rightarrow \mathfrak{F}$,! 17 (\Rightarrow I: 6,16)	i
$\neg Q \subset P$,! 18 (\neg I: 17)	i
$P \subset Q \Rightarrow \neg Q \subset P$,! 19 (\Rightarrow I: 2,18)	i
$(P \subset Q \Rightarrow \neg Q \subset P)$,! 20 ($(\)$ I: 19)	i
$\forall P \forall Q (P \subset Q \Rightarrow \neg Q \subset P)$! 21 (\forall I: 1,20)	i
\square		
! 58.		i
$\vdash \forall P \forall Q \forall R (P \subset Q \ \& \ Q \subseteq R \Rightarrow P \subset R)$		i
P, Q, R	,! 1 (Prem)	i
$P \subset Q \ \& \ Q \subseteq R$,! 2 (Prem)	i
$P \subset Q$,! 3 ($\&$ E: 2)	i
$Q \subseteq R$,! 4 ($\&$ E: 2)	i
$P \subseteq Q \ \& \ \neg P \equiv Q$,! 5 (\mathfrak{S} E: P49,3)	i
$P \subseteq Q$,! 6 ($\&$ E: 5)	i
$P \subseteq Q \ \& \ Q \subseteq R$,! 7 ($\&$ I: 4,6)	i
$(P \subseteq Q \ \& \ Q \subseteq R \Rightarrow P \subseteq R)$,! 8 (\forall E: P5)	i
$P \subseteq Q \ \& \ Q \subseteq R \Rightarrow P \subseteq R$,! 9 ($(\)$ E: 8)	i
$P \subseteq R$,! 10 (\Rightarrow E: 7,9)	i
$P \equiv R$,! 11 (Prem)	i
$(P \equiv R \Rightarrow R \subseteq P)$,! 12 (\forall E: P12)	i
$P \equiv R \Rightarrow R \subseteq P$,! 13 ($(\)$ E: 12)	i
$R \subseteq P$,! 14 (\Rightarrow E: 11,13)	i
$Q \subseteq R \ \& \ R \subseteq P$,! 15 ($\&$ I: 4,14)	i
$(Q \subseteq R \ \& \ R \subseteq P \Rightarrow Q \subseteq P)$,! 16 (\forall E: P5)	i
$Q \subseteq R \ \& \ R \subseteq P \Rightarrow Q \subseteq P$,! 17 ($(\)$ E: 16)	i

$Q \subseteq P$,! 18 (\Rightarrow E: 15,17)	i
$P \subseteq Q \ \& \ Q \subseteq P$,! 19 ($\&$ I: 6,18)	i
$(P \subseteq Q \ \& \ Q \subseteq P \Rightarrow P \equiv Q)$,! 20 (\forall E: P8)	i
$P \subseteq Q \ \& \ Q \subseteq P \Rightarrow P \equiv Q$,! 21 ($()$ E: 20)	i
$P \equiv Q$,! 22 (\Rightarrow E: 19,21)	i
$\neg P \equiv Q$,! 23 ($\&$ E: 5)	i
\mathfrak{F}	,! 24 (\mathfrak{F} I: 22,23)	i
$P \equiv R \Rightarrow \mathfrak{F}$,! 25 (\Rightarrow I: 11,24)	i
$\neg P \equiv R$,! 26 (\neg I: 25)	i
$P \subseteq R \ \& \ \neg P \equiv R$,! 27 ($\&$ I: 10,26)	i
$P \subset R$,! 28 (\mathfrak{S} I: P49,27)	i
$P \subset Q \ \& \ Q \subseteq R \Rightarrow P \subset R$,! 29 (\Rightarrow I: 2,28)	i
$(P \subset Q \ \& \ Q \subseteq R \Rightarrow P \subset R)$,! 30 ($()$ I: 29)	i
$\forall P \forall Q \forall R (P \subset Q \ \& \ Q \subseteq R \Rightarrow P \subset R)$! 31 (\forall I: 1,30)	i

□

! 59.

$\vdash \forall P \forall Q \forall R (P \subseteq Q \ \& \ Q \subset R \Rightarrow P \subset R)$		i
P, Q, R	,! 1 (Prem)	i
$P \subseteq Q \ \& \ Q \subset R$,! 2 (Prem)	i
$P \subseteq Q$,! 3 ($\&$ E: 2)	i
$Q \subset R$,! 4 ($\&$ E: 2)	i
$Q \subseteq R \ \& \ \neg Q \equiv R$,! 5 (\mathfrak{S} E: P49,4)	i
$Q \subseteq R$,! 6 ($\&$ E: 5)	i
$P \subseteq Q \ \& \ Q \subseteq R$,! 7 ($\&$ I: 3,6)	i
$(P \subseteq Q \ \& \ Q \subseteq R \Rightarrow P \subseteq R)$,! 8 (\forall E: P5)	i
$P \subseteq Q \ \& \ Q \subseteq R \Rightarrow P \subseteq R$,! 9 ($()$ E: 8)	i
$P \subseteq R$,! 10 (\Rightarrow E: 7,9)	i

$P \equiv R$,! 11 (Prem)	i
$(P \equiv R \Rightarrow R \subseteq P)$,! 12 ($\forall E$: P12)	i
$P \equiv R \Rightarrow R \subseteq P$,! 13 ($()E$: 12)	i
$R \subseteq P$,! 14 ($\Rightarrow E$: 11,13)	i
$R \subseteq P \ \& \ P \subseteq Q$,! 15 ($\&I$: 3,14)	i
$(R \subseteq P \ \& \ P \subseteq Q \Rightarrow R \subseteq Q)$,! 16 ($\forall E$: P5)	i
$R \subseteq P \ \& \ P \subseteq Q \Rightarrow R \subseteq Q$,! 17 ($()E$: 16)	i
$R \subseteq Q$,! 18 ($\Rightarrow E$: 15,17)	i
$Q \subseteq R \ \& \ R \subseteq Q$,! 19 ($\&I$: 6,18)	i
$(Q \subseteq R \ \& \ R \subseteq Q \Rightarrow Q \equiv R)$,! 20 ($\forall E$: P8)	i
$Q \subseteq R \ \& \ R \subseteq Q \Rightarrow Q \equiv R$,! 21 ($()E$: 20)	i
$Q \equiv R$,! 22 ($\Rightarrow E$: 19,21)	i
$\neg Q \equiv R$,! 23 ($\&E$: 5)	i
\mathcal{F}	,! 24 ($\mathcal{F}I$: 22,23)	i
$P \equiv R \Rightarrow \mathcal{F}$,! 25 ($\Rightarrow I$: 11,24)	i
$\neg P \equiv R$,! 26 ($\neg I$: 25)	i
$P \subseteq R \ \& \ \neg P \equiv R$,! 27 ($\&I$: 10,26)	i
$P \subset R$,! 28 ($\mathcal{S}I$: P49,27)	i
$P \subseteq Q \ \& \ Q \subset R \Rightarrow P \subset R$,! 29 ($\Rightarrow I$: 2,28)	i
$(P \subseteq Q \ \& \ Q \subset R \Rightarrow P \subset R)$,! 30 ($()I$: 29)	i
$\forall P \forall Q \forall R (P \subseteq Q \ \& \ Q \subset R \Rightarrow P \subset R)$! 31 ($\forall I$: 1,30)	i

□

! 60.

$\vdash \forall P \forall Q (P \subset Q \Rightarrow \exists x(Q[x] \ \& \ \neg P[x]))$		i
P, Q	,! 1 (Prem)	i
$P \subset Q$,! 2 (Prem)	i
$P \subseteq Q \ \& \ \neg P \equiv Q$,! 3 ($\mathcal{S}E$: P49,2)	i

$P \subseteq Q$,! 4 (&E: 3)	i
$\forall x(P[x] \Rightarrow Q[x])$,! 5 (\forall E: P1,4)	i
$\neg P \equiv Q$,! 6 (&E: 3)	i
$\neg \exists x(Q[x] \& \neg P[x])$,! 7 (Prem)	i
x	,! 8 (Prem)	i
$(P[x] \Rightarrow Q[x])$,! 9 (\forall E: 5)	i
$P[x] \Rightarrow Q[x]$,! 10 (()E: 9)	i
$Q[x]$,! 11 (Prem)	i
$\neg P[x]$,! 12 (Prem)	i
$Q[x] \& \neg P[x]$,! 13 (&I: 11,12)	i
$(Q[x] \& \neg P[x])$,! 14 (()I: 13)	i
$\exists x(Q[x] \& \neg P[x])$,! 15 (\exists I: 14)	i
\mathfrak{F}	,! 16 (\mathfrak{F} I: 7,15)	i
$\neg P[x] \Rightarrow \mathfrak{F}$,! 17 (\Rightarrow I: 12,16)	i
$\neg\neg P[x]$,! 18 (\neg I: 17)	i
$P[x]$,! 19 (\neg E: 18)	i
$Q[x] \Rightarrow P[x]$,! 20 (\Rightarrow I: 11,19)	i
$P[x] \Leftrightarrow Q[x]$,! 21 (\Leftrightarrow I: 10,20)	i
$(P[x] \Leftrightarrow Q[x])$,! 22 (()I: 21)	i
$\forall x(P[x] \Leftrightarrow Q[x])$,! 23 (\forall I: 8,22)	i
$P \equiv Q$,! 24 (\equiv I: P7, 23)	i
\mathfrak{F}	,! 25 (\mathfrak{F} I: 6,24)	i
$\neg \exists x(Q[x] \& \neg P[x]) \Rightarrow \mathfrak{F}$,! 26 (\Rightarrow I: 7,25)	i
$\neg\neg \exists x(Q[x] \& \neg P[x])$,! 27 (\neg I: 26)	i
$\exists x(Q[x] \& \neg P[x])$,! 28 (\neg E: 27)	i
$P \subset Q \Rightarrow \exists x(Q[x] \& \neg P[x])$,! 29 (\Rightarrow I: 2,28)	i
$(P \subset Q \Rightarrow \exists x(Q[x] \& \neg P[x]))$,! 30 (()I: 29)	i

$\forall P \forall Q (P \subset Q \Rightarrow \exists x(Q[x] \ \& \ \neg P[x]))$! 31 ($\forall I$: 1,30) i

□

! 61. P61 is an immediate corollary to P60. i

$\vdash \forall P \forall Q (P \subset Q \Rightarrow \exists x Q[x])$ i

P, Q ,! 1 (Prem) i

$P \subset Q$,! 2 (Prem) i

$(P \subset Q \Rightarrow \exists x(Q[x] \ \& \ \neg P[x]))$,! 3 ($\forall E$: P60) i

$P \subset Q \Rightarrow \exists x(Q[x] \ \& \ \neg P[x])$,! 4 ($(\Rightarrow)E$: 3) i

$\exists x(Q[x] \ \& \ \neg P[x])$,! 5 ($\Rightarrow E$: 2,4) i

$(Q[x] \ \& \ \neg P[x])$,! 6 ($\exists E$: 5) i

$Q[x] \ \& \ \neg P[x]$,! 7 ($(\Rightarrow)E$: 6) i

$Q[x]$,! 8 ($\&E$: 7) i

$\exists x Q[x]$,! 9 ($\exists I$: 8) i

$P \subset Q \Rightarrow \exists x Q[x]$,! 10 ($\Rightarrow I$: 2,9) i

$(P \subset Q \Rightarrow \exists x Q[x])$,! 11 ($(\Rightarrow)I$: 10) i

$\forall P \forall Q (P \subset Q \Rightarrow \exists x Q[x])$! 12 ($\forall I$: 1,11) i

□