# !CHAPTER 3

## SOME SIMPLE PROPOSITIONS¡

! The primary aim of this chapter is to prove various simple, fundamental logical facts, which may be needed in future proofs. It turns out that, except for the final four, the proofs do not appeal to prior propositions; instead the demonstrations are from first principles. However, several proofs could, without considerable extra length, be reconstructed so that they appealed to the previous proposition. For example there is a proof of P6, only slightly longer than the one actually used, which appeals to P5. ¡

! For completeness' sake, the symmetry and transitivity of equality are stated first, although they will never be used, since this would not result in a saving of steps.

For instance, consider symmetry of equality, namely

$$\forall x \forall y\ ( x = y \Rightarrow y = x ).$$

Suppose in the midst of a deduction **x** = **y** has already been proved. Symmetry of equality allows the deduction of **y** = **x** in three steps, as follows:

$( \mathbf{x} = \mathbf{y} \Rightarrow \mathbf{y} = \mathbf{x} )$ (∀E Prop)

$\mathbf{x} = \mathbf{y} \Rightarrow \mathbf{y} = \mathbf{x}$ (()E)

$\mathbf{y} = \mathbf{x}$ (⇒E)

It takes only two steps to use the rule of deduction =E:

$\mathbf{x} = \mathbf{x}$ (=E or =I)

$\mathbf{y} = \mathbf{x}$ (=E) ¡

### ! 1. Symmetry of Equality. ¡

$\vdash \forall x \forall y\ ( x = y \Rightarrow y = x )$ ¡

**x**,**y** ,! 1 (Prem) ¡

$\mathbf{x} = \mathbf{y}$ ,! 2 (Prem) ¡

! Since **x** = **y**, =E allows the substitution of **x** for **y** in any proposition already proved in the environment, including **x** = **y**. ¡

$\mathbf{x} = \mathbf{x}$ ,! 3 (=E: 2, 2) ¡

$\mathbf{y} = \mathbf{x}$ ,! 4 (=E: 2, 3) ¡

$\mathbf{x} = \mathbf{y} \Rightarrow \mathbf{y} = \mathbf{x}$ ,! 5 (⇒I: 2,4) ¡

$( \mathbf{x} = \mathbf{y} \Rightarrow \mathbf{y} = \mathbf{x} )$ ,! 6 (()I: 5) ¡

$\forall x \forall y\ ( x = y \Rightarrow y = x )$ ! 7 (∀I: 1,6) ¡

∎

### ! 2. Transitivity of Equality. ¡

$\vdash \forall x \forall y \forall z\ ( x = y \ \&\ y = z \Rightarrow x = z )$ ¡

**x**,**y**,**z** ,! 1 (Prem) ¡

$\mathbf{x} = \mathbf{y}\ \&\ \mathbf{y} = \mathbf{z}$ ,! 2 (Prem) ¡

**x** = **y** ,! 3 (&E: 2) ¡

**y** = **z** ,! 4 (&E: 2) ¡

**x** = **z** ,! 5 (=E: 3,4) ¡

**x** = **y** & **y** = **z** ⇒ **x** = **z** ,! 6 (⇒I: 2,5) ¡

( **x** = **y** & **y** = **z** ⇒ **x** = **z** ) ,! 7 (()I: 6) ¡

∀x∀y∀z ( x = y & y = z ⇒ x = z ) ! 8 (∀I: 2,7) ¡

▯

! **3. Symmetry of Inequality**. This theorem *will* be used. It is proved by *reductio ad absurdum*, i.e. by contradiction. ¡

⊦ ∀x∀y ( ¬ x = y ⇒ ¬ y = x ) ¡

**x**,**y** ,! 1 (Prem) ¡

¬ **x** = **y** ,! 2 (Prem) ¡

**y** = **x** ,! 3 (Prem) ¡

¬ **x** = **x** ,! 4 (=E: 2,3) ¡

**x** = **x** ,! 5 (=I) ¡

ℑ ,! 6 (ℑI: 4,5) ¡

**y** = **x** ⇒ ℑ ,! 7 (⇒I: 3,6) ¡

¬ **y** = **x** ,! 8 (¬I: 7) ¡

¬ **x** = **y** ⇒ ¬ **y** = **x** ,! 9 (⇒I: 2,8) ¡

( ¬ **x** = **y** ⇒ ¬ **y** = **x** ) ,! 10 (()I: 9) ¡

∀x∀y ( ¬ x = y ⇒ ¬ y = x ) ! 11 (∀I: 1,10) ¡

▯

! **4. Law of the Excluded Middle for Equality**. There will be other versions of the Law of the Excluded Middle stated below. Because there is no way to refer to propositions *qua* propositions, it is not possible to state a general Law of the Excluded Middle. Of course, for any statement P, one can prove P ∨ ¬ P, and such a proof follows the same technique used here to prove the LEM for Equality.

The inability to formulate a fully general Law of the Excluded Middle, and to formulate rules involving propositions in their full generality, could be solved by permitting and devising rules for "macros" or schemas. ¡

⊦ ∀x∀y ( x = y ∨ ¬ x = y ) ¡

**x**,**y** ,! 1 (Prem) ¡

¬ ( **x** = **y** ∨ ¬ **x** = **y** ) ,! 2 (Prem) ¡

**x** = **y** ,! 3 (Prem) ¡

**x** = **y** ∨ ¬ **x** = **y** ,! 4 (∨I: 3) ¡

( **x** = **y** ∨ ¬ **x** = **y** ) ,! 5 (()I: 4) ¡

ℑ ,! 6 (ℑI: 2,5) ¡

**x** = **y** ⇒ ℑ ,! 7 (⇒I: 3,6) ¡

¬ **x** = **y** ,! 8 (¬I: 7) ¡

**x** = **y** ∨ ¬ **x** = **y** ,! 9 (∨I: 8) ¡

( **x** = **y** ∨ ¬ **x** = **y** ) ,! 10 (()I: 9) ¡

ℑ ,! 11 (ℑI: 2,10) ¡

¬ ( **x** = **y** ∨ ¬ **x** = **y** ) ⇒ ℑ ,! 12 (⇒I: 2,11) ¡

¬¬ ( **x** = **y** ∨ ¬ **x** = **y** ) ,! 13 (¬I: 12) ¡

( **x** = **y** ∨ ¬ **x** = **y** ) ,! 14 (¬E: 13) ¡

∀x∀y ( x = y ∨ ¬ x = y ) ! 15 (∀I: 1,14) ¡

▯

! The next several propositions formalize various forms of the Process of Elimination, in the special case of equality and inequality. They are evidently particular cases of more general rules, but the general propositions are themselves not readily applicable, since equality is not represented in the same way as other two-place predicates.

For instance, it would obviously have been preferable to state the next proposition in its full generality, as

∀R∀S∀x∀y∀z∀a ( (R[x,y] ∨ S[z,a]) & ¬ R[x,y] ⇒ S[z,a] ),

and then to substitute '=' for 'R' and 'S', to get

∀x∀y∀z∀a ( (=[x,y] ∨ =[z,a]) & ¬ =[x,y] ⇒ =[z,a] ).

However, this would not have been useful, since FOEA does not relate =[x,y] and x = y, and the =E and =I rules are stated for the syntax 'x = y'.

Indeed, in order to state the idea in its full generality, it would be necessary again to use a "macro" or schema. ¡

! **5. Process of Elimination**, for two equals n1. ¡

⊦ ∀x∀y∀z∀a ( (x = y ∨ z = a) & ¬ x = y ⇒ z = a ) ¡

**x**,**y**,**z**,**a** ,! 1 (Prem) ¡

(**x** = **y** ∨ **z** = **a**) & ¬ **x** = **y** ,! 2 (Prem) ¡

(**x** = **y** ∨ **z** = **a**) ,! 3 (&E: 2) ¡

¬ **x** = **y** ,! 4 (&E: 2) ¡

**x** = **y** ∨ **z** = **a** ,! 5 (()E: 3) ¡

**x** = **y** ,! 6 (Prem) ¡

¬ **z** = **a** ,! 7 (Prem) ¡

ℱ ,! 8 (ℱI: 4,6) ¡

¬ **z** = **a** ⇒ ℱ ,! 9 (⇒I: 7,8) ¡

¬¬ **z** = **a** ,! 10 (¬I: 9) ¡

**z** = **a** ,! 11 (¬E: 10) ¡

**x** = **y** ⇒ **z** = **a** ,! 12 (⇒I: 6,11) ¡

**z** = **a** ,! 13 (Prem) ¡

**z** = **a** ⇒ **z** = **a** ,! 14 (⇒I: 13,13) ¡

**z** = **a** ,! 15 (∨E: 5,12,14) ¡

(**x** = **y** ∨ **z** = **a**) & ¬ **x** = **y** ⇒ **z** = **a** ,! 16 (⇒I: 2, 15) ¡

( (**x** = **y** ∨ **z** = **a**) & ¬ **x** = **y** ⇒ **z** = **a** ) ,! 17 (()I: 16) ¡

∀x∀y∀z∀a ( (x = y ∨ z = a) & ¬ x = y ⇒ z = a )
! 18 (∀I: 1,17) ¡

□

! **6. Process of Elimination**, for two equals n2. The proof is from first principles, although as is noted below, a slightly longer proof which appeals to the previous proposition could have been used. ¡

⊢ ∀x∀y∀z∀a ( (x = y ∨ z = a) & ¬ z = a ⇒ x = y ) ¡

**x**,**y**,**z**,**a** ,! 1 (Prem) ¡

(**x** = **y** ∨ **z** = **a**) & ¬ **z** = **a** ,! 2 (Prem) ¡

(**x** = **y** ∨ **z** = **a**) ,! 3 (&E: 2) ¡

¬ **z** = **a** ,! 4 (&E: 2) ¡

**x** = **y** ∨ **z** = **a** ,! 5 (()E: 3) ¡

**x** = **y** ,! 6 (Prem) ¡

**x** = **y** ⇒ **x** = **y** ,! 7 (⇒I: 6,6) ¡

**z** = **a** ,! 8 (Prem) ¡

¬ **x** = **y** ,! 9 (Prem) ¡

ℑ ,! 10 (ℑI: 6,9) ¡

¬ **x** = **y** ⇒ ℑ ,! 11 (⇒I: 9,10) ¡

¬¬ **x** = **y** ,! 12 (¬I: 11) ¡

**x** = **y** ,! 13 (¬E: 12) ¡

**z** = **a** ⇒ **x** = **y** ,! 14 (⇒I: 8,13) ¡

**x** = **y** ,! 15 (∨E: 5,7,14) ¡

(**x** = **y** ∨ **z** = **a**) & ¬ **z** = **a** ⇒ **x** = **y** ,! 16 (⇒I: 3,15) ¡

( (**x** = **y** ∨ **z** = **a**) & ¬ **z** = **a** ⇒ **x** = **y** ) ,! 17 (()I: 16) ¡

∀x∀y∀z∀a ( (x = y ∨ z = a) & ¬ z = a ⇒ x = y )
! 18 (∀I: 1,17) ¡

▯

! An alternative, slightly longer but more elegant, proof for P6, appeals to P5. Co-incident with the previous proof to step 5,from step 6 it proceeds as follows:

**x** = **y** , 6 (Prem)
**z** = **a** ∨ **x** = **y** , 7 (∨I: 6)
**x** = **y** ⇒ **z** = **a** ∨ **x** = **y** , 8 (⇒I: 6,7)
**z** = **a** , 9 (Prem)
**z** = **a** ∨ **x** = **y** , 10 (∨I: 9)
**x** = **y** ⇒ **z** = **a** ∨ **x** = **y** , 11 (⇒I: 9,10)
**z** = **a** ∨ **x** = **y** , 12 (∨E: 5,8,11)
(**z** = **a** ∨ **x** = **y**) , 13 (()I: 12)
(**z** = **a** ∨ **x** = **y**) & ¬ **z** = **a** , 14 (&I: 4,13)
( (**z** = **a** ∨ **x** = **y**) & ¬ **z** = **a** ⇒ **x** = **y** )
, 15 (∀E: P5)
(**z** = **a** ∨ **x** = **y**) & ¬ **z** = **a** ⇒ **x** = **y** , 16 (()E: 15)
**x** = **y** , 17 (⇒E: 14,16)
(**x** = **y** ∨ **z** = **a**) & ¬ **z** = **a** ⇒ **x** = **y** ,! 18 (⇒I: 3,17)
( (**x** = **y** ∨ **z** = **a**) & ¬ **z** = **a** ⇒ **x** = **y** ) ,! 19 (()I: 18)
∀x∀y∀z∀a ( (x = y ∨ z = a) & ¬ z = a ⇒ x = y )
! 20 (∀I: 1,19) ¡

! **7. Process of Elimination**, for two unequals n1. ¡

⊦ ∀x∀y∀z∀a ( (¬ x = y ∨ ¬ z = a) & x = y ⇒ ¬ z = a ) ¡

**x**,**y**,**z**,**a** ,! 1 (Prem) ¡

(¬ **x** = **y** ∨ ¬ **z** = **a**) & **x** = **y** ,! 2 (Prem) ¡

(¬ **x** = **y** ∨ ¬ **z** = **a**) ,! 3 (&E: 2) ¡

**x** = **y** ,! 4 (&E: 2) ¡

¬ **x** = **y** ∨ ¬ **z** = **a** ,! 5 (()E: 3) ¡

¬ **x** = **y** ,! 6 (Prem) ¡

**z** = **a** ,! 7 (Prem) ¡

ℱ ,! 8 (ℱI: 4,6) ¡

**z** = **a** ⇒ ℱ ,! 9 (⇒I: 7,8) ¡

¬ **z** = **a** ,! 10 (¬I: 9) ¡

¬ **x** = **y** ⇒ ¬ **z** = **a** ,! 11 (⇒I: 6,11) ¡

¬ **z** = **a** ,! 12 (Prem) ¡

¬ **z** = **a** ⇒ ¬ **z** = **a** ,! 13 (⇒I: 12,12) ¡

¬ **z** = **a** ,! 14 (∨E: 5,11,13) ¡

(¬ **x** = **y** ∨ ¬ **z** = **a**) & **x** = **y** ⇒ ¬ **z** = **a**,! 15 (⇒I: 2,14) ¡

( (¬ **x** = **y** ∨ ¬ **z** = **a**) & **x** = **y** ⇒ ¬ **z** = **a** )
,! 16 (()I: 15) ¡

∀x∀y∀z∀a ( (¬ x = y ∨ ¬ z = a) & x = y ⇒ ¬ z = a )
! 17 (∀I: 1,16) ¡

▯

! As with P6 from P5, P8 could also be proved by appealing to the previous proposition, P7, rather than from first principles. ¡

! **8. Process of Elimination**, for two unequals n2. ¡

⊢ ∀x∀y∀z∀a ( (¬ x = y ∨ ¬ z = a) & z = a ⇒ ¬ x = y ) ¡

**x**,**y**,**z**,**a** ,! 1 (Prem) ¡

(¬ **x** = **y** ∨ ¬ **z** = **a**) & **z** = **a** ,! 2 (Prem) ¡

(¬ **x** = **y** ∨ ¬ **z** = **a**) ,! 3 (&E: 2) ¡

**z** = **a** ,! 4 (&E: 2) ¡

¬ **x** = **y** ∨ ¬ **z** = **a** ,! 5 (()E: 3) ¡

¬ **z** = **a** ,! 6 (Prem) ¡

**x** = **y** ,! 7 (Prem) ¡

ℱ ,! 8 (ℱI: 4,6) ¡

**x** = **y** ⇒ ℑ ,! 9 (⇒I: 7,8) ¡

¬ **x** = **y** ,! 10 (¬I: 9) ¡

¬ **z** = **a** ⇒ ¬ **x** = **y** ,! 11 (⇒I: 6,10) ¡

¬ **x** = **y** ,! 12 (Prem) ¡

¬ **x** = **y** ⇒ ¬ **x** = **y** ,! 13 (⇒I: 12,12) ¡

¬ **x** = **y** ,! 14 (∨E: 5,11,13) ¡

(¬ **x** = **y** ∨ ¬ **z** = **a**) & **z** = **a** ⇒ ¬ **x** = **y**,! 15 (⇒I: 2,14) ¡

( (¬ **x** = **y** ∨ ¬ **z** = **a**) & **z** = **a** ⇒ ¬ **x** = **y** )
,! 16 (()I: 15) ¡

∀x∀y∀z∀a ( (¬ x = y ∨ ¬ z = a) & z = a ⇒ ¬ x = y )
! 17 (∀I: 1,16) ¡

□

! **9. Process of Elimination**, for three equals. ¡

⊢ ∀x∀y∀z∀a∀b∀c ( (x = y ∨ z = a ∨ b = c) & ¬ x = y
⇒ z = a ∨ b = c ) ¡

**x**,**y**,**z**,**a**,**b**,**c** ,! 1 (Prem) ¡

(**x** = **y** ∨ **z** = **a** ∨ **b** = **c**) & ¬ **x** = **y** ,! 2 (Prem) ¡

(**x** = **y** ∨ **z** = **a** ∨ **b** = **c**) ,! 3 (&E: 2) ¡

¬ **x** = **y** ,! 4 (&E: 2) ¡

**x** = **y** ∨ **z** = **a** ∨ **b** = **c** ,! 5 (()E: 3) ¡

**x** = **y** ,! 6 (Prem) ¡

¬ (**z** = **a** ∨ **b** = **c**) ,! 7 (Prem) ¡

ℑ ,! 8 (ℑI: 4,6) ¡

¬ (**z** = **a** ∨ **b** = **c**) ⇒ ℑ ,! 9 (⇒I: 7,8) ¡

¬¬ (**z** = **a** ∨ **b** = **c**) ,! 10 (¬I: 9) ¡

(**z** = **a** ∨ **b** = **c**) ,! 11 (¬E: 10) ¡

**z** = **a** ∨ **b** = **c** ,! 12 (()E: 11) ¡

**x** = **y** ⇒ **z** = **a** ∨ **b** = **c** ,! 13 (⇒I: 6,13) ¡

**z** = **a** ∨ **b** = **c** ,! 14 (Prem) ¡

z = a ∨ b = c ⇒ z = a ∨ b = c ,! 15 (⇒I: 14,14) ¡

z = a ∨ b = c ,! 16 (∨E: 5,13,15) ¡

(x = y ∨ z = a ∨ b = c) & ¬ x = y ⇒ z = a ∨ b = c
,! 17 (⇒I: 2,16) ¡

( (x = y ∨ z = a ∨ b = c) & ¬ x = y ⇒ z = a ∨ b = c )
,! 18 (()I: 17) ¡

∀x∀y∀z∀a∀b∀c ( (x = y ∨ z = a ∨ b = c) & ¬ x = y
⇒ z = a ∨ b = c )
! 19 (∀I: 1,18) ¡

▯

! **10. Process of Elimination**, for four equals. ¡

⊢ ∀x∀y∀z∀a∀b∀c∀d∀e ( (x = y ∨ z = a ∨ b = c ∨ d = e) & ¬ x = y
⇒ z = a ∨ b = c ∨ d = e ) ¡

x,y,z,a,b,c,d,e ,! 1 (Prem) ¡

(x = y ∨ z = a ∨ b = c ∨ d = e) & ¬ x = y
,! 2 (Prem) ¡

(x = y ∨ z = a ∨ b = c ∨ d = e) ,! 3 (&E: 2) ¡

¬ x = y ,! 4 (&E: 2) ¡

x = y ∨ z = a ∨ b = c ∨ d = e ,! 5 (()E: 3) ¡

x = y ,! 6 (Prem) ¡

¬ (z = a ∨ b = c ∨ d = e) ,! 7 (Prem) ¡

ℑ ,! 8 (ℑI: 4,6) ¡

¬ (z = a ∨ b = c ∨ d = e) ⇒ ℑ ,! 9 (⇒I: 7,8) ¡

¬¬ (z = a ∨ b = c ∨ d = e) ,! 10 (¬I: 9) ¡

(z = a ∨ b = c ∨ d = e) ,! 11 (¬E: 10) ¡

z = a ∨ b = c ∨ d = e ,! 12 (()E: 11) ¡

x = y ⇒ z = a ∨ b = c ∨ d = e ,! 13 (⇒I: 6,12) ¡

z = a ∨ b = c ∨ d = e ,! 14 (Prem) ¡

z = a ∨ b = c ∨ d = e ⇒ z = a ∨ b = c ∨ d = e
,! 15 (⇒I: 14,14) ¡

z = a ∨ b = c ∨ d = e ,! 16 (∨E: 5,13,15) ¡

(x = y ∨ z = a ∨ b = c ∨ d = e) & ¬ x = y
⇒ z = a ∨ b = c ∨ d = e
,! 17 (⇒I: 2,16) ¡

( (x = y ∨ z = a ∨ b = c ∨ d = e) & ¬ x = y
⇒ z = a ∨ b = c ∨ d = e )
,! 18 (()I: 17) ¡

∀x∀y∀z∀a∀b∀c∀d∀e ( (x = y ∨ z = a ∨ b = c ∨ d = e) & ¬ x = y
⇒ z = a ∨ b = c ∨ d = e )
! 19 (∀I: 1,18) ¡

▯

! **11. Process of Elimination**, for one-place predicates n1.
¡

⊢ ∀P∀Q∀x∀y ( (P[x] ∨ Q[y]) & ¬ P[x] ⇒ Q[y] ) ¡

P,Q,x,y ,! 1 (Prem) ¡

(P[x] ∨ Q[y]) & ¬ P[x] ,! 2 (Prem) ¡

(P[x] ∨ Q[y]) ,! 3 (&E: 2) ¡

¬ P[x] ,! 4 (&E: 2) ¡

P[x] ∨ Q[y] ,! 5 (()E: 3) ¡

P[x] ,! 6 (Prem) ¡

¬ Q[y] ,! 7 (Prem) ¡

ℱ ,! 8 (ℱI: 4,6) ¡

¬ Q[y] ⇒ ℱ ,! 9 (⇒I: 7,8) ¡

¬¬ Q[y] ,! 10 (¬I: 9) ¡

Q[y] ,! 11 (¬E: 10) ¡

P[x] ⇒ Q[y] ,! 12 (⇒I: 6,11) ¡

Q[y] ,! 13 (Prem) ¡

Q[y] ⇒ Q[y] ,! 14 (⇒I: 13,13) ¡

Q[y] ,! 15 (∨E: 5,12,14) ¡

(P[x] ∨ Q[y]) & ¬ P[x] ⇒ Q[y] ,! 16 (⇒I: 2,15) ¡

( (P[x] ∨ Q[y]) & ¬ P[x] ⇒ Q[y] ) ,! 17 (()I: 16) ¡

∀P∀Q∀x∀y ( (P[x] ∨ Q[y]) & ¬ P[x] ⇒ Q[y] )

! 18 (∀I: 1,17) ¡

▯

! As with P6 from P5 and P8 from P7, P12 could also be proved by appealing to the previous proposition, P11, rather than from first principles.
¡

! **12. Process of Elimination**, for one-place predicates n2.
¡

⊢ ∀P∀Q∀x∀y ( (P[x] ∨ Q[y]) & ¬ Q[y] ⇒ P[x] ) ¡

**P**,**Q**,**x**,**y** ,! 1 (Prem) ¡

(**P**[**x**] ∨ **Q**[**y**]) & ¬ **Q**[**y**] ,! 2 (Prem) ¡

(**P**[**x**] ∨ **Q**[**y**]) ,! 3 (&E: 2) ¡

¬ **Q**[**y**] ,! 4 (&E: 2) ¡

**P**[**x**] ∨ **Q**[**y**] ,! 5 (()E: 3) ¡

**P**[**x**] ,! 6 (Prem) ¡

**P**[**x**] ⇒ **P**[**x**] ,! 7 (⇒I: 6,6) ¡

**Q**[**y**] ,! 8 (Prem) ¡

¬ **P**[**x**] ,! 9 (Prem) ¡

ℱ ,! 10 (ℱI: 6,9) ¡

¬ **P**[**x**] ⇒ ℱ ,! 11 (⇒I: 9,10) ¡

¬¬ **P**[**x**] ,! 12 (¬I: 11) ¡

**P**[**x**] ,! 13 (¬E: 12) ¡

**Q**[**y**] ⇒ **P**[**x**] ,! 14 (⇒I: 8,13) ¡

**P**[**x**] ,! 15 (∨E: 5,7,14) ¡

(**P**[**x**] ∨ **Q**[**y**]) & ¬ **Q**[**y**] ⇒ **P**[**x**] ,! 16 (⇒I: 2,15) ¡

( (**P**[**x**] ∨ **Q**[**y**]) & ¬ **Q**[**y**] ⇒ **P**[**x**] ) ,! 17 (()I: 16) ¡

∀P∀Q∀x∀y ( (P[x] ∨ Q[y]) & ¬ Q[y] ⇒ P[x] )
! 18 (∀I: 1,17) ¡

▯

! **De Morgan's Laws (& and ∨ Form)** has several versions:
∀x∀y∀z∀a ( ¬ (x = y ∨ z = a) ⇔ ¬ x = y & ¬ z = a )
∀x∀y∀z∀a ( (x = y ∨ z = a) ⇔ ¬ (¬ x = y & ¬ z = a) )

∀x∀y∀z∀a ( (¬ x = y ∨ ¬ z = a) ⇔ ¬ (x = y & z = a) )
∀x∀y∀z∀a ( ¬ (¬ x = y ∨ ¬ z = a) ⇔ (x = y & z = a) )
Only the first will be proven, in two parts, here. ¡

**! 13. De Morgan's Law (& and ∨ Form) n1, First Half.** ¡

⊦ ∀x∀y∀z∀a ( ¬ (x = y ∨ z = a) ⇒ ¬ x = y & ¬ z = a ) ¡

**x,y,z,a** ,! 1 (Prem) ¡

¬ (**x** = **y** ∨ **z** = **a**) ,! 2 (Prem) ¡

**x** = **y** ,! 3 (Prem) ¡

**x** = **y** ∨ **z** = **a** ,! 4 (∨I: 3) ¡

(**x** = **y** ∨ **z** = **a**) ,! 5 (()I: 4) ¡

ℱ ,! 6 (ℱI: 2,5) ¡

**x** = **y** ⇒ ℱ ,! 7 (⇒I: 3,6) ¡

¬ **x** = **y** ,! 8 (¬I: 7) ¡

**z** = **a** ,! 9 (Prem) ¡

**x** = **y** ∨ **z** = **a** ,! 10 (∨I: 9) ¡

(**x** = **y** ∨ **z** = **a**) ,! 11 (()I: 10) ¡

ℱ ,! 12 (ℱI: 2,11) ¡

**z** = **a** ⇒ ℱ ,! 13 (⇒I: 9,12) ¡

¬ **z** = **a** ,! 14 (¬I: 13) ¡

¬ **x** = **y** & ¬ **z** = **a** ,! 15 (&I: 8,14) ¡

¬ (**x** = **y** ∨ **z** = **a**) ⇒ ¬ **x** = **y** & ¬ **z** = **a**
,! 16 (⇒I: 2,15) ¡

( ¬ (**x** = **y** ∨ **z** = **a**) ⇒ ¬ **x** = **y** & ¬ **z** = **a** )
,! 17 (()I: 16) ¡

∀x∀y∀z∀a ( ¬ (x = y ∨ z = a) ⇒ ¬ x = y & ¬ z = a )
! 18 (∀I: 1,17) ¡

▯

**! 14. De Morgan's Law (& and ∨ Form) n1, Second Half.** ¡

⊦ ∀x∀y∀z∀a ( ¬ x = y & ¬ z = a ⇒ ¬ (x = y ∨ z = a) ) ¡

**x,y,z,a** ,! 1 (Prem) ¡

¬ **x** = **y** & ¬ **z** = **a** ,! 2 (Prem) ¡

(**x** = **y** ∨ **z** = **a**) ,! 3 (Prem) ¡

**x** = **y** ∨ **z** = **a** ,! 4 (()E: 3) ¡

**x** = **y** ,! 5 (Prem) ¡

¬ **x** = **y** ,! 6 (&E: 2) ¡

ℑ ,! 7 (ℑI: 5,6) ¡

**x** = **y** ⇒ ℑ ,! 8 (⇒I: 5,7) ¡

**z** = **a** ,! 9 (Prem) ¡

¬ **z** = **a** ,! 10 (&E: 2) ¡

ℑ ,! 11 (ℑI: 9,10) ¡

**z** = **a** ⇒ ℑ ,! 12 (⇒I: 9,11) ¡

ℑ ,! 13 (∨E: 4,8,12 ) ¡

(**x** = **y** ∨ **z** = **a**) ⇒ ℑ ,! 14 (⇒I: 3,13) ¡

¬ (**x** = **y** ∨ **z** = **a**) ,! 15 (¬I: 14) ¡

¬ **x** = **y** & ¬ **z** = **a** ⇒ ¬ (**x** = **y** ∨ **z** = **a**),! 16 (⇒I: 2,15) ¡

( ¬ **x** = **y** & ¬ **z** = **a** ⇒ ¬ (**x** = **y** ∨ **z** = **a**) )
,! 17 (()I: 16) ¡

∀x∀y∀z∀a ( ¬ x = y & ¬ z = a ⇒ ¬ (x = y ∨ z = a) )
! 18 (∀I: 1,17) ¡

! **15. Law of the Excluded Middle for One-Place Predicates.**
¡

⊢ ∀P∀x ( P[x] ∨ ¬ P[x] ) ¡

**P**,**x** ,! 1 (Prem) ¡

¬ (**P**[**x**] ∨ ¬ **P**[**x**]) ,! 2 (Prem) ¡

**P**[**x**] ,! 3 (Prem) ¡

**P**[**x**] ∨ ¬ **P**[**x**] ,! 4 (∨I: 3) ¡

(**P**[**x**] ∨ ¬ **P**[**x**]) ,! 5 (()I: 4) ¡

ℑ ,! 6 (ℑI: 2,5) ¡

**P**[**x**] ⇒ ℑ ,! 7 (⇒I: 3,6) ¡

¬ **P**[**x**] ,! 8 (¬I: 7) ¡

**P**[**x**] ∨ ¬ **P**[**x**] ,! 9 (∨I: 8) ¡

(**P**[**x**] ∨ ¬ **P**[**x**]) ,! 10 (()I: 9) ¡

ℑ ,! 11 (ℑI: 2,10) ¡

¬ (**P**[**x**] ∨ ¬ **P**[**x**]) ⇒ ℑ ,! 12 (⇒I: 11) ¡

¬¬ (**P**[**x**] ∨ ¬ **P**[**x**]) ,! 13 (¬I: 12) ¡

(**P**[**x**] ∨ ¬ **P**[**x**]) ,! 14 (¬E: 13) ¡

∀P∀x ( P[x] ∨ ¬ P[x] ) ! 15 (∀I: 1,14) ¡

▯

! **16. Leibniz's Law of Indiscernibles, First Half.** The second half will be proved in chapter II.8, once it has been established that, for every thing, there exists a predicate which is only satisfied by that thing. ¡

⊦ ∀x∀y ( x = y ⇒ ∀P(P[x] ⇔ P[y]) ) ¡

**x**,**y** ,! 1 (Prem) ¡

**x** = **y** ,! 2 (Prem) ¡

**P** ,! 3 (Prem) ¡

**P**[**x**] ,! 4 (Prem) ¡

**P**[**y**] ,! 5 (=E: 2,4) ¡

**P**[**x**] ⇒ **P**[**y**] ,! 6 (⇒I: 4,5) ¡

**P**[**y**] ,! 7 (Prem) ¡

**P**[**x**] ,! 8 (=E: 2,7) ¡

**P**[**y**] ⇒ **P**[**x**] ,! 9 (⇒I: 7,8) ¡

**P**[**x**] ⇔ **P**[**y**] ,! 10 (⇔I: 6,9) ¡

(**P**[**x**] ⇔ **P**[**y**]) ,! 11 (()I: 10) ¡

∀P(P[**x**] ⇔ P[**y**]) ,! 12 (∀I: 3,11) ¡

**x** = **y** ⇒ ∀P(P[**x**] ⇔ P[**y**]) ,! 13 (⇒I: 2,12) ¡

( **x** = **y** ⇒ ∀P(P[**x**] ⇔ P[**y**]) ) ,! 14 (()I: 13) ¡

∀x∀y ( x = y ⇒ ∀P(P[x] ⇔ P[y]) ) ! 15 (∀I: 1,14) ¡

▯

! Leibniz's Law will never be used, because =E is much simpler and more direct. The following proposition, however, does save steps.

¡

! **17.** ¡

⊦ ∀P∀x∀y ( P[x] & ¬ P[y] ⇒ ¬ x = y ) ¡

**P**,**x**,**y** ,! 1 (Prem) ¡

**P**[**x**] & ¬ **P**[**y**] ,! 2 (Prem) ¡

**P**[**x**] ,! 3 (&E: 2) ¡

¬ **P**[**y**] ,! 4 (&E: 2) ¡

**x** = **y** ,! 5 (Prem) ¡

**P**[**y**] ,! 6 (=E: 3,5) ¡

ℑ ,! 7 (ℑI: 4,6) ¡

**x** = **y** ⇒ ℑ ,! 8 (⇒I: 5,7) ¡

¬ **x** = **y** ,! 9 (¬I: 8) ¡

**P**[**x**] & ¬ **P**[**y**] ⇒ ¬ **x** = **y** ,! 10 (⇒I: 2,9) ¡

( **P**[**x**] & ¬ **P**[**y**] ⇒ ¬ **x** = **y** ) ,! 11 (()I: 10) ¡

∀P∀x∀y ( P[x] & ¬ P[y] ⇒ ¬ x = y ) ! 12 (∀I: 1,12) ¡

▯

! **18. Law of the Excluded Middle for Two-Place Predicates.** ¡

⊦ ∀R∀x∀y ( R[x,y] ∨ ¬ R[x,y] ) ¡

**R**,**x**,**y** ,! 1 (Prem) ¡

¬ ( **R**[**x**,**y**] ∨ ¬ **R**[**x**,**y**] ) ,! 2 (Prem) ¡

**R**[**x**,**y**] ,! 3 (Prem) ¡

**R**[**x**,**y**] ∨ ¬ **R**[**x**,**y**] ,! 4 (∨I: 3) ¡

(**R**[**x**,**y**] ∨ ¬ **R**[**x**,**y**]) ,! 5 (()I: 4) ¡

ℑ ,! 6 (ℑI: 2,5) ¡

**R**[**x**,**y**] ⇒ ℑ ,! 7 (⇒I: 3,6) ¡

¬ **R**[**x**,**y**] ,! 8 (¬I: 7) ¡

**R**[**x**,**y**] ∨ ¬ **R**[**x**,**y**] ,! 9 (∨I: 3,8) ¡

( **R**[**x**,**y**] ∨ ¬ **R**[**x**,**y**] ) ,! 10 (()I: 9) ¡

ℑ ,! 11 (ℑI: 2,10) ¡

¬ ( **R**[**x**,**y**] ∨ ¬ **R**[**x**,**y**] ) ⇒ ℱ ,! 12 (⇒I: 2,11) ¡

¬¬ ( **R**[**x**,**y**] ∨ ¬ **R**[**x**,**y**] ) ,! 13 (¬I: 12) ¡

( **R**[**x**,**y**] ∨ ¬ **R**[**x**,**y**] ) ,! 14 (¬E: 13) ¡

∀R∀x∀y ( R[x,y] ∨ ¬ R[x,y] ) ! 15 (∀I: 1,14) ¡

▯

! The next several propositions (P19-P28) concern **De Morgan's Laws** for Existential and Universal Quantifiers. They are linked to De Morgan's Laws for & and ∨ because over a finite domain Existential Quantification is essentially disjunction, and Universal Quantification conjunction. ¡

! **19. De Morgan's Law** for Quantifiers v1, First Half. ¡

⊦ ∀P ( ∀x P[x] ⇒ ¬ ∃x ¬ P[x] ) ¡

**P** ,! 1 (Prem) ¡

∀x **P**[x] ,! 2 (Prem) ¡

∃x ¬ **P**[x] ,! 3 (Prem) ¡

¬ **P**[**a**] ,! 4 (∃E: 3) ¡

**P**[**a**] ,! 5 (∀E: 2) ¡

ℱ ,! 6 (ℱI: 4,5) ¡

∃x ¬ **P**[x] ⇒ ℱ ,! 7 (⇒I: 3,6) ¡

¬ ∃x ¬ **P**[x] ,! 8 (¬I: 7) ¡

∀x **P**[x] ⇒ ¬ ∃x ¬ **P**[x] ,! 9 (⇒I: 3,8) ¡

( ∀x **P**[x] ⇒ ¬ ∃x ¬ **P**[x] ) ,! 10 (()I: 9) ¡

∀P ( ∀x P[x] ⇒ ¬ ∃x ¬ P[x] ) ! 11 (∀I: 1,10) ¡

▯

! **20. De Morgan's Law** for Quantifiers v2, First Half. ¡

⊦ ∀P ( ∃x P[x] ⇒ ¬ ∀x ¬ P[x] ) ¡

**P** ,! 1 (Prem) ¡

∃x **P**[x] ,! 2 (Prem) ¡

∀x ¬ **P**[x] ,! 3 (Prem) ¡

**P**[**a**] ,! 4 (∃E: 2) ¡

¬ **P**[**a**] ,! 5 (∀E: 3) ¡

ℑ ,! 6 (ℑI: 4,5) ¡

∀x ¬ **P**[x] ⇒ ℑ ,! 7 (⇒I: 3,6) ¡

¬ ∀x ¬ **P**[x] ,! 8 (¬I: 7) ¡

∃x **P**[x] ⇒ ¬ ∀x ¬ **P**[x] ,! 9 (⇒I: 2,8) ¡

( ∃x **P**[x] ⇒ ¬ ∀x ¬ **P**[x] ) ,! 10 (()I: 9) ¡

∀P ( ∃x P[x] ⇒ ¬ ∀x ¬ P[x] ) ! 11 (∀I: 1,10) ¡

▯

! **21. De Morgan's Law** for Quantifiers v3, First Half. ¡

⊢ ∀P ( ∀x ¬ P[x] ⇒ ¬ ∃x P[x] ) ¡

**P** ,! 1 (Prem) ¡

∀x ¬ **P**[x] ,! 2 (Prem) ¡

∃x **P**[x] ,! 3 (Prem) ¡

**P**[**a**] ,! 4 (∃E: 3) ¡

¬ **P**[**a**] ,! 5 (∀E: 2) ¡

ℑ ,! 6 (ℑI: 4,5) ¡

∃x **P**[x] ⇒ ℑ ,! 7 (⇒I: 3,6) ¡

¬ ∃x **P**[x] ,! 8 (¬I: 7) ¡

∀x ¬ **P**[x] ⇒ ¬ ∃x **P**[x] ,! 9 (⇒I: 2,8) ¡

( ∀x ¬ **P**[x] ⇒ ¬ ∃x **P**[x] ) ,! 10 (()I: 9) ¡

∀P ( ∀x ¬ P[x] ⇒ ¬ ∃x P[x] ) ! 11 (∀I: 1,10) ¡

▯

! **22. De Morgan's Law** for Quantifiers v4, First Half. ¡

⊢ ∀P ( ∃x ¬ P[x] ⇒ ¬ ∀x P[x] ) ¡

**P** ,! 1 (Prem) ¡

∃x ¬ **P**[x] ,! 2 (Prem) ¡

∀x **P**[x] ,! 3 (Prem) ¡

¬ **P**[**a**] ,! 4 (∃E: 2) ¡

**P**[**a**] ,! 5 (∀E: 3) ¡

ℑ ,! 6 (ℑI: 4,5) ¡

∀x **P**[x] ⇒ ℑ ,! 7 (⇒I: 3,6) ¡

¬ ∀x **P**[x] ,! 8 (¬I: 7) ¡

∃x ¬ **P**[x] ⇒ ¬ ∀x **P**[x] ,! 9 (⇒I: 2,8) ¡

( ∃x ¬ **P**[x] ⇒ ¬ ∀x **P**[x] ) ,! 10 (()I: 9) ¡

∀P ( ∃x ¬ P[x] ⇒ ¬ ∀x P[x] ) ! 11 (∀I: 1,10) ¡

▯

! **23. De Morgan's Law** for Quantifiers v4, Second Half. ¡

⊢ ∀P ( ¬ ∀x P[x] ⇒ ∃x ¬ P[x] ) ¡

**P** ,! 1 (Prem) ¡

¬ ∀x **P**[x] ,! 2 (Prem) ¡

¬ ∃x ¬ **P**[x] ,! 3 (Prem) ¡

**x** ,! 4 (Prem) ¡

¬ **P**[**x**] ,! 5 (Prem) ¡

∃x ¬**P**[x] ,! 6 (∃I: 5) ¡

ℑ ,! 7 (ℑI: 3,6) ¡

¬ **P**[**x**] ⇒ ℑ ,! 8 (⇒I: 5,7) ¡

¬¬ **P**[**x**] ,! 9 (¬I: 8) ¡

**P**[**x**] ,! 10 (¬E: 9) ¡

∀x **P**[x] ,! 11 (∀I: 4,10) ¡

ℑ ,! 12 (ℑI: 2,11) ¡

¬ ∃x ¬ **P**[x] ⇒ ℑ ,! 13 (⇒I: 3,12) ¡

¬¬ ∃x ¬ **P**[x] ,! 14 (¬I: 13) ¡

∃x ¬ **P**[x] ,! 15 (¬E: 14) ¡

¬ ∀x **P**[x] ⇒ ∃x ¬ **P**[x] ,! 16 (⇒I: 2,15) ¡

( ¬ ∀x **P**[x] ⇒ ∃x ¬ **P**[x] ) ,! 17 (()I: 16) ¡

∀P ( ¬ ∀x P[x] ⇒ ∃x ¬ P[x] ) ! 18 (∀I: 1,17) ¡

▯

! **24. De Morgan's Law** for Quantifiers v3, Second Half. ¡

⊢ ∀P ( ¬ ∃x P[x] ⇒ ∀x ¬ P[x] ) ¡

**P** ,! 1 (Prem) ¡

¬ ∃x **P**[x] ,! 2 (Prem) ¡

¬ ∀x ¬ **P**[x] ,! 3 (Prem) ¡

**x** ,! 4 (Prem) ¡

**P**[**x**] ,! 5 (Prem) ¡

∃x **P**[x] ,! 6 (∃I: 5) ¡

𝔉 ,! 7 (𝔉I: 2,6) ¡

**P**[**x**] ⇒ 𝔉 ,! 8 (⇒I: 5,7) ¡

¬ **P**[**x**] ,! 9 (¬I: 8) ¡

∀x ¬ **P**[x] ,! 10 (∀I: 4,9) ¡

𝔉 ,! 11 (𝔉I: 3,10) ¡

¬ ∀x ¬ **P**[x] ⇒ 𝔉 ,! 12 (⇒I: 3,11) ¡

¬¬ ∀x ¬ **P**[x] ,! 13 (¬I: 12) ¡

∀x ¬ **P**[x] ,! 14 (¬E: 13) ¡

¬ ∃x **P**[x] ⇒ ∀x ¬ **P**[x] ,! 15 (⇒I: 2,14) ¡

( ¬ ∃x **P**[x] ⇒ ∀x ¬ **P**[x] ) ,! 16 (()I: 15) ¡

∀P ( ¬ ∃x P[x] ⇒ ∀x ¬ P[x] ) ! 17(∀I: 1,16) ¡

▯

! **25. De Morgan's Law** for Quantifiers v2, Second Half. ¡

⊢ ∀P ( ¬ ∀x ¬ P[x] ⇒ ∃x P[x] ) ¡

**P** ,! 1 (Prem) ¡

¬ ∀x ¬ **P**[x] ,! 2 (Prem) ¡

¬ ∃x **P**[x] ,! 3 (Prem) ¡

( ¬ ∃x **P**[x] ⇒ ∀x ¬ **P**[x] ) ,! 4 (∀E: P24) ¡

¬ ∃x **P**[x] ⇒ ∀x ¬ **P**[x] ,! 5 (()E: 4) ¡

∀x ¬ **P**[x] ,! 6 (⇒E: 3,5) ¡

𝔉 ,! 7 (𝔉I: 2,6) ¡

¬ ∃x **P**[x] ⇒ 𝔉 ,! 8 (⇒I: 3,7) ¡

¬¬ ∃x **P**[x] ,! 9 (¬I: 8) ¡

∃x **P**[x] ,! 10 (¬E: 9) ¡

¬ ∀x ¬ **P**[x] ⇒ ∃x **P**[x] ,! 11 (⇒I: 2,10) ¡

( ¬ ∀x ¬ **P**[x] ⇒ ∃x **P**[x] ) ,! 12 (()I: 11) ¡

∀P ( ¬ ∀x ¬ P[x] ⇒ ∃x P[x] ) ! 13 (∀I: 1,12) ¡

□

! **26. De Morgan's Law** for Quantifiers v1, Second Half. ¡

⊢ ∀P ( ¬ ∃x ¬ P[x] ⇒ ∀x P[x] ) ¡

**P** ,! 1 (Prem) ¡

¬ ∃x ¬ **P**[x] ,! 2 (Prem) ¡

¬ ∀x **P**[x] ,! 3 (Prem) ¡

( ¬ ∀x **P**[x] ⇒ ∃x ¬ **P**[x] ) ,! 4 (∀E: P23) ¡

¬ ∀x **P**[x] ⇒ ∃x ¬ **P**[x] ,! 5 (()E: 4) ¡

∃x ¬ **P**[x] ,! 6 (⇒E: 3,5) ¡

𝔉 ,! 7 (𝔉I: 2,6) ¡

¬ ∀x **P**[x] ⇒ 𝔉 ,! 8 (⇒I: 3,7) ¡

¬¬ ∀x **P**[x] ,! 9 (¬I: 8) ¡

∀x **P**[x] ,! 10 (¬E: 9) ¡

¬ ∃x ¬ **P**[x] ⇒ ∀x **P**[x] ,! 11 (⇒I: 2,10) ¡

( ¬ ∃x ¬ **P**[x] ⇒ ∀x **P**[x] ) ,! 12 (()I: 11) ¡

∀P ( ¬ ∃x ¬ P[x] ⇒ ∀x P[x] ) ! 13 (∀I: 1,12) ¡

□

! **27. De Morgan's Law** for Quantifiers v1. ¡

⊢ ∀P ( ∀x P[x] ⇔ ¬ ∃x ¬ P[x] ) ¡

**P** ,! 1 (Prem) ¡

( ∀x **P**[x] ⇒ ¬ ∃x ¬ **P**[x] ) ,! 2 (∀E: P19) ¡

∀x **P**[x] ⇒ ¬ ∃x ¬ **P**[x] ,! 3 (()E: 3) ¡

( ¬ ∃x ¬ **P**[x] ⇒ ∀x **P**[x] ) ,! 4 (∀E: P26) ¡

¬ ∃x ¬ **P**[x] ⇒ ∀x **P**[x] ,! 5 (()E: 4) ¡

∀x **P**[x] ⇔ ¬ ∃x ¬ **P**[x] ,! 6 (⇔I: 3,5) ¡

( ∀x **P**[x] ⇔ ¬ ∃x ¬ **P**[x] ) ,! 7 (()I: 6) ¡

∀P ( ∀x P[x] ⇔ ¬ ∃x ¬ P[x] ) ! 8 (∀I: 1,7) ¡

▯

! **28. De Morgan's Law** for Quantifiers v2. ¡

⊢ ∀P ( ∃x P[x] ⇔ ¬ ∀x ¬ P[x] ) ¡

**P** ,! 1 (Prem) ¡

( ∃x **P**[x] ⇒ ¬ ∀x ¬ **P**[x] ) ,! 2 (∀E: P20) ¡

∃x **P**[x] ⇒ ¬ ∀x ¬ **P**[x] ,! 3 (()E: 2) ¡

( ¬ ∀x ¬ **P**[x] ⇒ ∃x **P**[x] ) ,! 4 (∀E: P25) ¡

¬ ∀x ¬ **P**[x] ⇒ ∃x **P**[x] ,! 5 (()E: 4) ¡

∃x **P**[x] ⇔ ¬ ∀x ¬ **P**[x] ,! 6 (⇔I: 3,5) ¡

( ∃x **P**[x] ⇔ ¬ ∀x ¬ **P**[x] ) ,! 7 (()I: 6) ¡

∀P ( ∃x P[x] ⇔ ¬ ∀x ¬ P[x] ) ! 8 (∀I: 1,7) ¡

▯